Online Appendix of "Large Firm Dynamics and the Business Cycle"

Vasco M. Carvalho and Basile Grassi
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B Proof Appendix

In this proof appendix, we first prove two intermediate results that (i) describe the stationary distribution for a finite S (Lemma 1), and, (ii) describe the limit of the steady-state value of the wage w and the entry/exit threshold s^* when S goes to infinity (Lemma 2). This is given in Appendix B.1. We then prove Proposition 1, giving the value and policy functions of an incumbent firm at the steady-state, in Appendix B.2. We then prove Corollary 2 giving the stationary distribution when $S \to \infty$. In Appendix B.4, we prove Proposition 2, giving the ergodic behavior of the firm productivity distribution for the case without entry and exit. In Appendix B.5, we state and prove a general theorem that extends Theorem 2 to the case with entry and exit. We then prove Proposition 3. We then find the asymptotic value of the ratio between the number of incumbents and the number of potential entrants, when the former goes to infinity (Appendix B.7). This intermediate result will be used in the the proof of Propositions 4 and 5 in Appendix B.8. Finally, we prove Proposition 6 that solve for the value and policy function under Assumption 3. This last proof involves two intermediate results, Lemma 3 and 4.

B.1 Preliminary Results

Lemma 1 *For a given S, if (i) the entrant distribution is Pareto (i.e $G_s = K_e(\varphi^s)^{-\delta_e}$) and (ii) the productivity process follows Gibrat's law (Assumption 1) with parameters a and c on the grid defined by φ, then the stationary distribution (i.e when $\mathbb{V}ar_t\epsilon_{t+1} = 0$) is:*
For $s^* \leq s \leq S$:

$$\mu_s = \mathbb{P}\{\varphi = \varphi^s\} = MK_eC_1\left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + MK_eC_2(\varphi^s)^{-\delta_e} + MK_eC_3$$

and $\mu_{s^*-1} = a\left(\mu_{s^*} + MK_e\left(\varphi^{s^*}\right)^{-\delta_e}\right)$ and $\mu_s = 0$ for $s < s^* - 1$.
Where $\delta = \frac{\log(a/c)}{\log(\varphi)}$ and C_1, C_2, C_3 are constants, independent of s, and where
$C_1 = \frac{c\left(a(\varphi^{-\delta_e})^{S+2} - a(\varphi^{-\delta_e})^{s^*} - c(\varphi^{-\delta_e})^{S+3} + c(\varphi^{-\delta_e})^{s^*}\right)}{a(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e}-c)}$, $C_2 = \frac{(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c)}{(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c)+c)}$ *and* $C_3 = \frac{-(\varphi^{-\delta_e})^{S+1}}{(1-\varphi^{-\delta_e})(a-c)}$.

Proof: To find the stationary distribution of the Markovian process described by the transition matrix P, we need to solve for μ in $\mu = (P_t^*)'(\mu + MG)$ where P is given by assumption 1 and where μ is the $(S \times 1)$ vector $(\mu_1, \ldots, \mu_S)'$. For simplicity, we assume $M = 1$.

The matrix equation $\mu = (P_t^*)'(\mu + MG)$ can be equivalently written as the following system of equations:

For $s < s^* - 1$:
$$\mu_s = 0 \tag{20}$$

For $s = s^* - 1$:
$$\mu_{s^*-1} = a(\mu_{s^*} + G_{s^*}) \tag{21}$$

For $s = s^*$:
$$\mu_{s^*} = b(\mu_{s^*} + G_{s^*}) + a(\mu_{s^*+1} + G_{s^*+1}) \tag{22}$$

For $s = S$:
$$\mu_S = c(\mu_{S-1} + G_{S-1}) + (b + c)(\mu_S + G_S) \tag{23}$$

For $s^* + 1 \leq s \leq S - 1$:

$$\mu_s = c(\mu_{s-1} + G_{s-1}) + b(\mu_s + G_s) + a(\mu_{s+1} + G_{s+1}) \tag{24}$$

The system of Equations 22, 23 and 24 gives a linear second order difference equation with two boundary conditions. The system has a exogenous term given by the distribution of entrants G. For this system, we define the associated homogeneous system by the same equations with $G_s = 0, \forall s$. To solve for a linear second order difference equation, we follow four steps: (i) Solve for the general solution of the homogeneous system; these solutions are parametrized by two constants (ii) Find one particular solution for the full system (iii) The general solution of the full system is then given by the sum of the general solution of the homogeneous system and the particular solution we have found (iv) Solve for the undetermined coefficient using the boundary conditions.

The recurrence equation of the homogeneous system is equivalent to $c\mu_{s-1} - (a+c)\mu_s + a\mu_{s+1} = 0$ since $b = 1 - a - c$. To find the general solution of this equation, let us solve for the root of the polynomial $aX^2 - (a+c)X + c$. This polynomial is equal to $a(X - c/a)(X - 1)$ and thus its roots are $r_1 = c/a$ and 1. The general solution of the homogeneous system associated to Equation 24 is then $\mu_s = A(c/a)^s + B$ where A and B are constants.

Using the form of the entrant distribution $G_s = K_e(\varphi^{-\delta_e})^s$, and assuming that $\varphi^{-\delta_e} \neq \frac{a}{c}$, a particular solution is $K_e \frac{a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c}{a(\varphi^{-\delta_e})^2 - (a+c)\varphi^{-\delta_e} + c}(\varphi^{-\delta_e})^s$.

The general solution of the second order linear difference equation is then

$$A(c/a)^s + B + K_e \frac{a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c}{a(\varphi^{-\delta_e})^2 - (a+c)\varphi^{-\delta_e} + c}(\varphi^{-\delta_e})^s$$

By substituting this general solution in the boundary condition 22 and 23, we find

$$A = K_e \left(\frac{c}{a}\right)^{-s^*} \frac{c\left(a(\varphi^{-\delta_e})^{S+2} - a(\varphi^{-\delta_e})^{s^*} - c(\varphi^{-\delta_e})^{S+3} + c(\varphi^{-\delta_e})^{s^*}\right)}{a(1 - \varphi^{-\delta_e})(a - c)(a\varphi^{-\delta_e} - c)} \quad \text{and} \quad B = K_e \frac{-(\varphi^{-\delta_e})^{S+1}}{(1 - \varphi^{-\delta_e})(a - c)}$$

Since the s^{th} productivity level is φ^s, then $s = \frac{\log \varphi^s}{\log \varphi}$ and thus $\left(\frac{c}{a}\right)^s = (\varphi^s)^{-\frac{\log a/c}{\log \varphi}}$. Let us define $\delta = \frac{\log a/c}{\log \varphi}$. The expression of the stationary distribution is then:

$$\mu_s = K_e C_1 \left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + K_e C_2 (\varphi^s)^{-\delta_e} + K_e C_3 \tag{25}$$

for $s^* \leq s \leq S$. The value of μ_{s^*-1} is given by 21 and $\forall s < s^* - 1, \mu_s = 0$. □

Lemma 2 *The limits $\overline{s^*}$ and $\overline{w}$ of s^* and w when S goes to infinity satisfy $\overline{w} = \left(\alpha^{\frac{1}{1-\alpha}} A^\infty\right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}}$ where*

$$\frac{A}{M} \underset{S\to\infty}{\longrightarrow} A^\infty := a(\varphi^{\overline{s^*}-1})^{\frac{1}{1-\alpha}} \left((\varphi^{\delta_e} - 1)C_1^\infty + (\varphi^{\delta_e} - 1)(C_2 + 1)\left(\varphi^{\overline{s^*}}\right)^{-\delta_e}\right) + (\varphi^{\delta_e} - 1)C_1^\infty \frac{\left(\varphi^{\frac{1}{1-\alpha}}\right)^{\overline{s^*}}}{1 - \varphi^{-\delta + \frac{1}{1-\alpha}}} + (\varphi^{\delta_e} - 1)C_2 \frac{\left(\varphi^{-\delta_e + \frac{1}{1-\alpha}}\right)^{\overline{s^*}}}{1 - \varphi^{-\delta_e + \frac{1}{1-\alpha}}}$$

and $C_2 = \frac{(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c)}{(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c) + c)}$, as defined in Lemma 1 and $C_1^\infty = \frac{c}{a} \frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{(1 - \varphi^{-\delta_e})(c - a\varphi^{-\delta_e})}$

Proof: To show this lemma, let us first note that $w = \left(\alpha^{\frac{1}{1-\alpha}}\frac{A}{M}\right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}}$ and let us take the limit of $\frac{A}{M}$ when S goes to infinity. For a given S, let us look at the expression of A:

$$
\begin{aligned}
A =& \sum_{s=1}^{S}(\varphi^s)^{\frac{1}{1-\alpha}}\mu_s \\
=& (\varphi^{s^*-1})^{\frac{1}{1-\alpha}}\mu_{s^*-1} + \sum_{s=s^*}^{S}(\varphi^s)^{\frac{1}{1-\alpha}}\mu_s \\
=& (\varphi^{s^*-1})^{\frac{1}{1-\alpha}} a\left(MK_eC_1 + MK_eC_2\left(\varphi^{s^*}\right)^{-\delta_e} + MK_eC_3 + MK_e(\varphi^{s^*})^{-\delta_e}\right) \\
&+ \sum_{s=s^*}^{S}(\varphi^s)^{\frac{1}{1-\alpha}}\left(MK_eC_1\left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + MK_eC_2\left(\varphi^s\right)^{-\delta_e} + MK_eC_3\right)
\end{aligned}
$$

By dividing both sides by M, we get

$$
\begin{aligned}
\frac{A}{M} =& a(\varphi^{s^*-1})^{\frac{1}{1-\alpha}}\left(K_eC_1 + K_eC_2\left(\varphi^{s^*}\right)^{-\delta_e} + K_eC_3 + K_e(\varphi^{s^*})^{-\delta_e}\right) \\
&+ K_eC_1\left(\varphi^{s^*}\right)^{\delta}\sum_{s=s^*}^{S}\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^s + K_eC_2\sum_{s=s^*}^{S}\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^s + K_eC_3\sum_{s=s^*}^{S}(\varphi^{\frac{1}{1-\alpha}})^s \\
=& a(\varphi^{s^*-1})^{\frac{1}{1-\alpha}}\left(K_eC_1 + K_eC_2\left(\varphi^{s^*}\right)^{-\delta_e} + K_eC_3 + K_e(\varphi^{s^*})^{-\delta_e}\right) \\
&+ K_eC_1\left(\varphi^{s^*}\right)^{\delta}\frac{\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^{s^*} - \left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^{S+1}}{1-\varphi^{-\delta+\frac{1}{1-\alpha}}} + K_eC_2\frac{\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{s^*} - \left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{S+1}}{1-\varphi^{-\delta_e+\frac{1}{1-\alpha}}} \\
&+ K_eC_3\frac{(\varphi^{\frac{1}{1-\alpha}})^{s^*} - (\varphi^{\frac{1}{1-\alpha}})^{S+1}}{1-\varphi^{\frac{1}{1-\alpha}}}
\end{aligned}
$$

Since $\varphi > 1$, $\delta(1-\alpha) > 1$ and $\delta_e(1-\alpha) > 1$, we have that $-\frac{\delta_e}{\delta} + \frac{1}{\delta(1-\alpha)} < 0$ and $-1 + \frac{1}{\delta(1-\alpha)} < 0$. This implies that both $\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^S$ and $\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^S$ converge to zero when S goes to infinity. We also have that

$$
C_3(\varphi^{\frac{1}{1-\alpha}})^S = \frac{-(\varphi^{-\delta_e})^{S+1}}{(1-\varphi^{-\delta_e})(a-c)}(\varphi^{\frac{1}{1-\alpha}})^S = \frac{-\varphi^{-\delta_e}(\varphi^{-\delta_e+\frac{1}{1-\alpha}})^S}{(1-\varphi^{-\delta_e})(a-c)} \underset{S\to\infty}{\longrightarrow} 0
$$

Putting these results together yields

$$
\begin{aligned}
\frac{A}{M} \underset{S\to\infty}{\longrightarrow} A^{\infty} :=& a(\varphi^{s^*-1})^{\frac{1}{1-\alpha}}\left((\varphi^{\delta_e}-1)C_1^{\infty} + (\varphi^{\delta_e}-1)(C_2+1)\left(\varphi^{s^*}\right)^{-\delta_e}\right) \\
&+ (\varphi^{\delta_e}-1)C_1^{\infty}\frac{\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s^*}}{1-\varphi^{-\delta+\frac{1}{1-\alpha}}} + (\varphi^{\delta_e}-1)C_2\frac{\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{s^*}}{1-\varphi^{-\delta_e+\frac{1}{1-\alpha}}}
\end{aligned}
$$

□

B.2 Proof of Proposition 1

In this section we prove Proposition 1. We first solve for the value and the policy function for the general case of a finite S and then present the simpler special case - given in the main text - when S goes to infinity.

Instantaneous profit: It is easy to show that instantaneous profit is equal to

$$\pi^*(\mu, \varphi^s) = (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f$$

Note that this is a function of μ through the equilibrium wage w. In the stationary equilibrium this wage is fixed. In the following we will drop the notation μ whenever no confusion arises from this.

Bellman equation: In the stationary equilibrium, the Bellman equation is given by

$$V_s = (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f + \beta \max\{0, aV_{s-1} + bV_s + cV_{s+1}\}$$

where $V_s = V(\mu, \varphi^s)$. The policy function of this problem follows a threshold rule: there exist a s^* such that

$$V_s = (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f + \beta\left(aV_{s-1} + bV_s + cV_{s+1}\right) \qquad \text{for } s \geq s^*$$

$$V_s = (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f \qquad \text{for } s \leq s^* - 1$$

For $s \geq s^*$: Let us first look at the case when $s \geq s^*$. We want to solve for the following second order linear difference equation:

$$aV_{s-1} + \left(1 - a - c - \frac{1}{\beta}\right) V_s + cV_{s+1} = \frac{c_f}{\beta} - (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\beta} \tag{26}$$

which is associated with the homogeneous equation

$$aV_{s-1} + \left(1 - a - c - \frac{1}{\beta}\right) V_s + cV_{s+1} = 0 \tag{27}$$

This homogeneous equation is associated with the polynomial $cX^2 + \left(1 - a - c - \frac{1}{\beta}\right) X + a$ which has discriminant $\Delta = \left(1 - a - c - \frac{1}{\beta}\right)^2 - 4ca = \left(\frac{\beta-1}{\beta}\right)^2 + (a-c)^2 + 2(a+c)\frac{1-\beta}{\beta} > 0$. Thus, this polynomial has two real roots:

$$r_1 = \frac{(a + c + \frac{1}{\beta} - 1) + \sqrt{\Delta}}{2c} \quad \text{and} \quad r_2 = \frac{(a + c + \frac{1}{\beta} - 1) - \sqrt{\Delta}}{2c}$$

Since $a - c + \frac{1}{\beta} - 1 > 0$ it is trivial to show that $r_2 < 1 < r_1$. The general solution of the homogeneous Equation 27 is

$$V_s = K_1 r_1^s + K_2 r_2^s$$

where K_1 and K_2 are (for now) undetermined constants.

To find the general solution of the Equation 26, we need to find a particular solution of this equation. A particular solution of Equation 26 is

$$V_s = -\frac{c_f}{1-\beta} + (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}$$

where $\rho = a\varphi^{\frac{-1}{1-\alpha}} + b + c\varphi^{\frac{1}{1-\alpha}}$.

The general solution of Equation 26 takes the following form

$$V_s^{GS} = K_1 r_1^s + K_2 r_2^s - \frac{c_f}{1-\beta} + (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}$$

where K_1 and K_2 are constants to be solved for. To solve for these constants we use the boundary conditions.

At $s = s^*$**,** the value function of a firms satisfies

$$aV_{s^*-1} + \left(1 - a - c - \frac{1}{\beta}\right) V^{GS}_{s^*} + cV^{GS}_{s^*+1} = \frac{c_f}{\beta} - \left(\varphi^{s^*}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\beta}$$

with $V_{s^*-1} = \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f$. Note that V^{GS}_s also satisfies

$$aV^{GS}_{s^*-1} + \left(1 - a - c - \frac{1}{\beta}\right) V^{GS}_{s^*} + cV^{GS}_{s^*+1} = \frac{c_f}{\beta} - \left(\varphi^{s^*}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\beta}$$

It follows that $V^{GS}_{s^*-1} = V_{s^*-1}$, which yields

$$K_1 r_1^{s^*-1} + K_2 r_2^{s^*-1} - \frac{c_f}{1-\beta} + \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta} = \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f$$

After rearranging terms we get

$$K_1 r_1^{s^*-1} + K_2 r_2^{s^*-1} = \beta \frac{c_f}{1-\beta} - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta} \tag{28}$$

At $s = S$**,** the value function at level φ^S, V_S, satisfies

$$aV^{GS}_{S-1} + (1 - a - c + c)V_S = \frac{1}{\beta} V_S + \frac{c_f}{\beta} - \frac{1-\alpha}{\beta} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S$$

Solving for V_S yields

$$V_S = \frac{1}{1 - \frac{1}{\beta} - a} \left(\frac{c_f}{\beta} - \frac{1-\alpha}{\beta} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S - aV^{GS}_{S-1}\right)$$

which implies

$$V_S = \frac{1}{1 - \frac{1}{\beta} - a} \left(\frac{c_f}{\beta} - \frac{1-\alpha}{\beta} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S - a(K_1 r_1^{S-1} + K_2 r_2^{S-1}) + a\frac{c_f}{1-\beta} - a\left(\varphi^{S-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}\right)$$

$$V_S = \frac{1}{1 - \frac{1}{\beta} - a} \left(c_f \left(\frac{1}{\beta} + a\frac{1}{1-\beta}\right) - (1-\alpha) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{1}{\beta} + a\frac{\varphi^{\frac{-1}{1-\alpha}}}{1-\beta\rho}\right) - a(K_1 r_1^{S-1} + K_2 r_2^{S-1})\right)$$

$$V_S = \frac{1}{1 - \frac{1}{\beta} - a} \left(c_f \left(\frac{\frac{1}{\beta} - 1 + a}{1-\beta}\right) - (1-\alpha) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{1}{\beta} + a\frac{\varphi^{\frac{-1}{1-\alpha}}}{1-\beta\rho}\right) - a(K_1 r_1^{S-1} + K_2 r_2^{S-1})\right)$$

$$V_S = \frac{1}{1 - \frac{1}{\beta} - a} \left(c_f \left(\frac{\frac{1}{\beta} - 1 + a}{1-\beta}\right) - (1-\alpha) \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\frac{1}{\beta} - \rho + a\varphi^{\frac{-1}{1-\alpha}}}{1-\beta\rho}\right) - a(K_1 r_1^{S-1} + K_2 r_2^{S-1})\right)$$

$$V_S = \frac{-c_f}{1-\beta} - \frac{1-\alpha}{1-\beta\rho} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\frac{1}{\beta} - \rho + a\varphi^{\frac{-1}{1-\alpha}}}{1 - \frac{1}{\beta} - a}\right) - a(K_1 r_1^{S-1} + K_2 r_2^{S-1})$$

At $s = S - 1$**,** we have

$$aV^{GS}_{S-2} + \left(1 - a - c - \frac{1}{\beta}\right) V^{GS}_{S-1} + cV_S = \frac{c_f}{\beta} - \left(\varphi^{S-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\beta}$$

but, at the same time

$$aV_{S-2}^{GS} + \left(1 - a - c - \frac{1}{\beta}\right) V_{S-1}^{GS} + cV_S^{GS} = \frac{c_f}{\beta} - \left(\varphi^{S-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{\beta}$$

it follows that $V_S = V_S^{GS}$ and thus

$$\frac{-c_f}{1-\beta} - \frac{1-\alpha}{1-\beta\rho} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\frac{1}{\beta} - \rho + a\varphi^{\frac{-1}{1-\alpha}}}{1 - \frac{1}{\beta} - a}\right) - a(K_1 r_1^{S-1} + K_2 r_2^{S-1}) = K_1 r_1^S + K_2 r_2^S - \frac{c_f}{1-\beta} + \left(\varphi^S\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}$$

$\Leftrightarrow$

$$-\frac{1-\alpha}{1-\beta\rho} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\frac{1}{\beta} - \rho + a\varphi^{\frac{-1}{1-\alpha}}}{1 - \frac{1}{\beta} - a}\right) - \left(\varphi^S\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta} = K_1 r_1^S + K_2 r_2^S + a(K_1 r_1^{S-1} + K_2 r_2^{S-1})$$

$\Leftrightarrow$

$$(1 + ar_1^{-1})K_1 r_1^S + (1 + ar_2^{-1})K_2 r_2^S = -\frac{1-\alpha}{1-\beta\rho} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\frac{1}{\beta} - \rho + a\varphi^{\frac{-1}{1-\alpha}}}{1 - \frac{1}{\beta} - a} + 1\right)$$

which yields

$$(1 + ar_1^{-1})K_1 r_1^S + (1 + ar_2^{-1})K_2 r_2^S = -\frac{1-\alpha}{1-\beta\rho} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{a(\varphi^{\frac{-1}{1-\alpha}} - 1) + 1 - \rho}{1 - \frac{1}{\beta} - a}\right) \tag{29}$$

Solving for K_1 and K_2: Equations 28 and 29 form a system of two equations in two unknowns. Solving this system gives K_1 and K_2 and thus the full solution of the incumbent's value function over the state space Φ. Let us rewrite the system of Equations 28 and 29 as

$$K_1 r_1^{s^*-1} + K_2 r_2^{s^*-1} = A - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} B$$
$$(1 + ar_1^{-1})K_1 r_1^S + (1 + ar_2^{-1})K_2 r_2^S = -\kappa \left(\varphi^{\frac{1}{1-\alpha}}\right)^S B$$

where $A = \beta\frac{c_f}{1-\beta}$, $B = \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}$ and $\kappa = \frac{a(\varphi^{\frac{-1}{1-\alpha}}-1)+1-\rho}{1-\frac{1}{\beta}-a}$. It is obvious to show that

$$K_1(s^*) = \frac{(1 + ar_2^{-1})r_2^{S-s^*+1} \left(A - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} B\right) + \kappa \left(\varphi^{\frac{1}{1-\alpha}}\right)^S B}{(1 + ar_2^{-1})r_2^{S-s^*+1} r_1^{s^*-1} - (1 + ar_1^{-1})r_1^S}$$

$$K_2(s^*) = \frac{(1 + ar_1^{-1})r_1^{S-s^*+1} \left(A - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} B\right) + \kappa \left(\varphi^{\frac{1}{1-\alpha}}\right)^S B}{(1 + ar_1^{-1})r_1^{S-s^*+1} r_2^{s^*-1} - (1 + ar_2^{-1})r_2^S}$$

or, after substituting the expression for A, B and κ,

$$K_1(s^*, w) = \frac{(1 + ar_2^{-1})r_2^{S-s^*+1} \left(\beta\frac{c_f}{1-\beta} - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}\right) + \frac{a(\varphi^{\frac{-1}{1-\alpha}}-1)+1-\rho}{1-\frac{1}{\beta}-a} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}}{(1 + ar_2^{-1})r_2^{S-s^*+1} r_1^{s^*-1} - (1 + ar_1^{-1})r_1^S}$$

$$K_2(s^*, w) = \frac{(1 + ar_1^{-1})r_1^{S-s^*+1} \left(\beta\frac{c_f}{1-\beta} - \beta\rho \left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}\right) + \frac{a(\varphi^{\frac{-1}{1-\alpha}}-1)+1-\rho}{1-\frac{1}{\beta}-a} \left(\varphi^{\frac{1}{1-\alpha}}\right)^S \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta}}{(1 + ar_1^{-1})r_1^{S-s^*+1} r_2^{s^*-1} - (1 + ar_2^{-1})r_2^S}$$

Note that both K_1 and K_2 are also function of the wage and the threshold s^*. It follows that the unique solution of the Bellman equation is

$$V_s = \begin{cases} K_1(s^*,w)r_1^s + K_2(s^*,w)r_2^s - \frac{c_f}{1-\beta} + (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} \frac{1-\alpha}{1-\rho\beta} & \text{for } s \geq s^* \\ (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f & \text{for } s \leq s^*-1 \end{cases}$$

Solving for s^*: Note that by definition s^* is the smallest integer such that $aV_{s^*-1} + bV_{s^*} + cV_{s^*+1} \geq 0$ (i.e that $aV_{s^*-2} + bV_{s^*-1} + cV_{s^*} < 0$). Note also that

$$ar_1^{s-1} + br_1^s + cr_1^{s+1} = \frac{r_1^s}{\beta}$$
$$ar_2^{s-1} + br_2^s + cr_2^{s+1} = \frac{r_2^s}{\beta}$$
$$a\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s-1} + b\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s} + c\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s+1} = \rho\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s}$$

by definition of r_1, r_2 and ρ. Using the above equations, it is easy to show that

$$aV_{s^*-1} + bV_{s^*} + cV_{s^*+1} = \frac{1}{\beta}\left(K_1(s^*,w)r_1^{s^*} + K_2(s^*,w)r_2^{s^*}\right) - \frac{c_f}{1-\beta} + \rho\left(\varphi^{s^*}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{1-\rho\beta}$$

Solving for $\widetilde{s}^*$ such that $\frac{1}{\beta}\left(K_1(\widetilde{s}^*,w)r_1^{\widetilde{s}^*} + K_2(\widetilde{s}^*,w)r_2^{\widetilde{s}^*}\right) - \frac{c_f}{1-\beta} + \rho\left(\varphi^{\widetilde{s}^*}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{1-\rho\beta} = 0$ implies that $s^* = \lceil \widetilde{s}^* \rceil$. This completes the characterization of the solution of the Bellman equation. In order to obtain a more intuitive expression, we now turn to the special case where $S \to \infty$.

Solution of the Bellman when $S \to \infty$: Since $r_1 > \varphi^{\frac{1}{1-\alpha}} > 1 > r_2$ (we know that $\varphi^{\frac{1}{1-\alpha}} > 1 > r_2$ and we have assumed that $r_1 > \varphi^{\frac{1}{1-\alpha}}$ i.e φ is small enough), it is easy to show that

$$K_1(s^*) = \frac{(1+ar_2^{-1})r_2^{S-s^*+1}\left(A - \beta\rho\left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}}B\right) + \kappa\left(\varphi^{\frac{1}{1-\alpha}}\right)^S B}{(1+ar_2^{-1})r_2^{S-s^*+1}r_1^{s^*-1} - (1+ar_1^{-1})r_1^S} \underset{S\to\infty}{\longrightarrow} 0$$

$$K_2(s^*) = \frac{(1+ar_1^{-1})r_1^{S-s^*+1}\left(A - \beta\rho\left(\varphi^{s^*-1}\right)^{\frac{1}{1-\alpha}}B\right) + \kappa\left(\varphi^{\frac{1}{1-\alpha}}\right)^S B}{(1+ar_1^{-1})r_1^{S-s^*+1}r_2^{s^*-1} - (1+ar_2^{-1})r_2^S} \underset{S\to\infty}{\longrightarrow} \frac{A - \beta\rho\left(\varphi^{\overline{s^*}-1}\right)^{\frac{1}{1-\alpha}}\overline{B}}{r_2^{\overline{s^*}-1}}$$

It follows that for $s \geq \overline{s^*}$

$$V_s^{S=\infty} = \left(A - \beta\rho\left(\varphi^{\overline{s^*}-1}\right)^{\frac{1}{1-\alpha}}\overline{B}\right)r_2^{s-\overline{s^*}+1} - \frac{c_f}{1-\beta} + \left(\varphi^{\frac{1}{1-\alpha}}\right)^s\left(\frac{\alpha}{\overline{w}}\right)^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{1-\rho\beta}$$

where $A = \beta\frac{c_f}{1-\beta}$ and $\overline{B} = \left(\frac{\alpha}{\overline{w}}\right)^{\frac{\alpha}{1-\alpha}}\frac{1-\alpha}{1-\rho\beta}$ and $\overline{w}$ and $\overline{s^*}$ are the limits of, respectively, w and s^* when S goes to infinity. After substituting the expression of A and $\overline{B}$ and rearranging terms, the solution of the Bellman equation is, for all s:

$$V_s^{S=\infty} = \frac{-c_f}{1-\beta}\left(1 - \beta r_2^{[s-\overline{s^*}+1]^+}\right) + \frac{1-\alpha}{1-\rho\beta}\left(\frac{\alpha}{\overline{w}}\right)^{\frac{\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s\left(1 - \rho\beta\left(\frac{r_2}{\varphi^{\frac{1}{1-\alpha}}}\right)^{[s-\overline{s^*}+1]^+}\right)$$

where $[x]^+ = \frac{|x|+x}{2} = \max(x,0)$.

Solving for s^* when $S \to \infty$: Following the same steps as in the case $S < \infty$, it is easy to show that, for $s \geq \overline{s^*}$,

$$aV_{s-1} + bV_s + cV_{s+1} = \frac{-c_f}{1-\beta}\left(1 - r_2^{s-\overline{s^*}+1}\right) + \frac{1-\alpha}{1-\rho\beta}\left(\frac{\alpha}{\overline{w}}\right)^{\frac{\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s\left(\rho - \rho\left(\frac{r_2}{\varphi^{\frac{1}{1-\alpha}}}\right)^{[s-\overline{s^*}+1]^+}\right)$$

and thus, for $s = \overline{s^*}$,

$$aV_{\overline{s^*}-1} + bV_{\overline{s^*}} + cV_{\overline{s^*}+1} = \frac{-c_f}{1-\beta}(1-r_2) + \frac{1-\alpha}{1-\rho\beta}\left(\frac{\alpha}{\overline{w}}\right)^{\frac{\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{\overline{s^*}}\rho\left(1 - \frac{r_2}{\varphi^{\frac{1}{1-\alpha}}}\right)$$

It follows that $aV_{\overline{s^*}-1} + bV_{\overline{s^*}} + cV_{\overline{s^*}+1} \geq 0$ is equivalent to

$$\overline{s^*} \geq (1-\alpha)\frac{\log\left[\frac{c_f(1-r_2)(1-\rho\beta)}{\rho(1-\beta)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(1-r_2\varphi^{\frac{-1}{1-\alpha}})}\right]}{\log\varphi} + \alpha\frac{\log\overline{w}}{\log\varphi}$$

Since $\overline{s^*}$ is the smallest integer such that this inequality is satisfied, it follows that

$$\overline{s^*} = \left\lceil (1-\alpha)\frac{\log\left[\frac{c_f(1-r_2)(1-\rho\beta)}{\rho(1-\beta)(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}(1-r_2\varphi^{\frac{-1}{1-\alpha}})}\right]}{\log\varphi} + \alpha\frac{\log\overline{w}}{\log\varphi}\right\rceil$$

which complete the proof. □

B.3 Proof of Corollary 2

In this appendix, we prove that the productivity stationary distribution is a mixture of two distributions: (i) the stationary distribution associated with the Markovian firm-level productivity process and (ii) the distribution of entrants. These are weighted by the constants K_1 and K_2, respectively. Formally, we show that $K_1 = -\frac{c}{a}\frac{(\varphi^{\delta_e}-1)(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(a\varphi^{-\delta_e}-c)}$ and $K_2 = \frac{(\varphi^{\delta_e}-1)(a(\varphi^{-\delta_e})^2+b\varphi^{-\delta_e}+c)}{a(\varphi^{-\delta_e})^2-\varphi^{-\delta_e}(a+c)+c}(\varphi^{\overline{s^*}})^{-\delta_e}$. In the corollary in the main text, we only reported the value of the stationary productivity distribution for productivity levels above the entry/exit thresholds. In this appendix, for completeness, we describe this distribution over the full idiosyncratic state-space. We then show that:

$$\hat{\mu}_s = \begin{cases} -\frac{c}{a}\frac{(\varphi^{\delta_e}-1)(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(a\varphi^{-\delta_e}-c)}\left(\frac{\varphi^s}{\varphi^{\overline{s^*}}}\right)^{-\delta} + \frac{(\varphi^{\delta_e}-1)\left(a(\varphi^{-\delta_e})^2+b\varphi^{-\delta_e}+c\right)}{a(\varphi^{-\delta_e})^2-\varphi^{-\delta_e}(a+c)+c}(\varphi^s)^{-\delta_e} & \text{if } s \geq \overline{s^*} \\ a\left(\varphi^{\delta_e}-1\right)\left(\frac{-c/a}{(1-\varphi^{-\delta_e})(a\varphi^{-\delta_e}-c)} + \frac{a(\varphi^{-\delta_e})^2+b\varphi^{-\delta_e}+c}{a(\varphi^{-\delta_e})^2-(a+c)\varphi^{-\delta_e}+c}+1\right)(\varphi^{\overline{s^*}})^{-\delta_e} & \text{if } i = \overline{s^*}-1 \\ 0 & \text{if } s < \overline{s^*}-1 \end{cases}$$

with $\delta = \frac{\log(a/c)}{\log(\varphi)}$.

The proof of this corollary builds on the result of Lemma 1 and then takes the limit of this distribution when the maximum level of productivity goes to infinity.

We first find the limit of constants K_e, C_1, C_2 and C_3 as the number of productivity bins S goes to infinity. After finding these limits, we take the limit of Equation 25 in the previous lemma.

Let us first describe the asymptotic behavior of K_e. Recall that the entrant distribution sums to one.[37]

$$1=\sum_{s=1}^{S} G_s = K_e \sum_{s=1}^{S} (\varphi^s)^{-\delta_e} = K_e \sum_{s=1}^{S} \left(\varphi^{-\delta_e}\right)^s = K_e \frac{\varphi^{-\delta_e}-\left(\varphi^{-\delta_e}\right)^{S+1}}{1-\varphi^{-\delta_e}}$$

Rearranging terms, it follows that

$$K_e = \frac{1-\varphi^{-\delta_e}}{\varphi^{-\delta_e}-\left(\varphi^{-\delta_e}\right)^{S+1}}$$

Since $\varphi > 1$ and $\delta_e, \delta > 0$ we have $\left(\varphi^{-\delta_e}\right)^S \underset{S\to\infty}{\longrightarrow} 0$ by applying these results to the expression for K_e, it follows that $K_e \underset{S\to\infty}{\longrightarrow} \varphi^{\delta_e}-1$. Let us now focus on the asymptotic behavior of C_3, C_2 and C_1. From Lemma 1, we have $C_3 = \frac{-(\varphi^{-\delta_e})^{S+1}}{(1-\varphi^{-\delta_e})(a-c)} \underset{S\to\infty}{\longrightarrow} 0$. We also have that

$$C_2 := \frac{(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c)}{(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c)+c)}$$

which is independent of S.

Finally, we have

$$\begin{aligned} C_1 =& \frac{c\left(a(\varphi^{-\delta_e})^{S+2} - a(\varphi^{-\delta_e})^{s^*} - c(\varphi^{-\delta_e})^{S+3} + c(\varphi^{-\delta_e})^{s^*}\right)}{a(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e}-c)} \\ \underset{S\to\infty}{\longrightarrow}& \frac{c\left(-a(\varphi^{-\delta_e})^{\overline{s^*}} + c(\varphi^{-\delta_e})^{\overline{s^*}}\right)}{a(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e}-c)} = \frac{c}{a}\frac{-(a-c)(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e}-c)} \end{aligned}$$

and therefore

$$C_1 \underset{S\to\infty}{\longrightarrow} C_1^\infty := \frac{c}{a}\frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(c-a\varphi^{-\delta_e})}$$

We have just found the limit of K_e, C_1, C_2 and C_3 when S goes to infinity. We then apply these results to the stationary distribution by taking S to infinity. According to Lemma 1, we have for $s^* \leq s$:

$$\frac{\mu_s}{M} = K_e C_1 \left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + K_e C_2 (\varphi^s)^{-\delta_e} + K_e C_3$$

We have just shown that when S goes to infinity, the stationary distribution is given by:

$$\frac{\mu_s}{M} = \left(\varphi^{\delta_e}-1\right)\frac{c}{a}\frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(c-a\varphi^{-\delta_e})}\left(\frac{\varphi^s}{\varphi^{\overline{s^*}}}\right)^{-\delta} + \left(\varphi^{\delta_e}-1\right)\frac{(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e}+c)}{(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c)+c)}(\varphi^s)^{-\delta_e}$$

□

B.4 Proof of Proposition 2

Proposition 2 claims that for the no entry and exit case and under Assumption 1, the unconditional mean of μ_t is given by

$$\mathbb{E}\left[\mu_{s,t}\right] = \mu_s = N\frac{1-\varphi^{-\delta}}{\varphi^{-\delta}(1-(\varphi^S)^{-\delta})}\left(\varphi^s\right)^{-\delta}$$

where $\delta = \frac{\log(a/c)}{\log(\varphi)}$. Furthermore, the unconditional variance-covariance matrix of μ_t is

$$\mathbb{V}ar\left[\mu_t\right] = \sum_{k=0}^{\infty}(P')^k\left(\sum_{s=1}^{S}\mu_s W_s\right)P^k$$

[37] The way we define the model, we assume that G sums to one. We also assume that the number of potential entrants in bin s is MG_s, so that the total number of potential entrants is M.

where P is the transition matrix for firm-level productivity, and, $W_s = diag(P_{s,.}) - P'_{s,.}P_{s,.}$ where $P_{s,.}$ denotes the s^{th}-row of the transition matrix P in Assumption 1. Where for $1 < s < S$, $W_s = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$ with $\Sigma = \begin{pmatrix} a(1-a) & -ab & -ac \\ -ab & b(1-b) & -bc \\ -ac & -bc & c(1-c) \end{pmatrix}$, while $W_1 = \begin{pmatrix} \Sigma^{(1)} & 0 \\ 0 & 0 \end{pmatrix}$ with $\Sigma^{(1)} = \begin{pmatrix} c(1-c) & -c(1-c) \\ -c(1-c) & c(1-c) \end{pmatrix}$, and, $W_S = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma^{(S)} \end{pmatrix}$ with $\Sigma^{(S)} = \begin{pmatrix} a(1-a) & -a(1-a) \\ -a(1-a) & a(1-a) \end{pmatrix}$.

Proof:

Let us define $f_{t+1}^{k,s}$ as the number of firms in state k at $t+1$ that were in state s at t. Under Assumption 1, it is easy to show that, for $1 < s < S$, $f_{t+1}^{k,s} = 0$ for both $k > s+1$ and $k < s-1$. Similarly, we have $f_{t+1}^{k,1} = 0$ for $k > 2$ and $f_{t+1}^{k,S}$ for $k < S-1$. It is easy to see that

$$\begin{aligned} \mu_{1,t+1} &= f_{t+1}^{1,1} + f_{t+1}^{1,2} && \text{for } s = 1 \\ \mu_{s,t+1} &= f_{t+1}^{s,s-1} + f_{t+1}^{s,s} + f_{t+1}^{s,s+1} && \text{for } 1 < s < S \\ \mu_{S,t+1} &= f_{t+1}^{S,S-1} + f_{t+1}^{S,S} && \text{for } s = S \end{aligned}$$

As in the proof of Theorem 1, the vector $f_{t+1}^{\cdot,s} = (f_{t+1}^{s-1,s}, f_{t+1}^{s,s}, f_{t+1}^{s+1,s})'$ is distributed according to a multinomial distribution with number of trials $\mu_{s,t}$ and probability of events $(a,b,c)'$. As the number of firms in productivity state s becomes large, we can approximate this multinomial distribution with a normal distribution (see Severini 2005, p377 example 12.7). It follows that, for $1 < s < S$, we have:

$$f_{t+1}^{\cdot,s} = \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\mu_{s,t}\begin{pmatrix} a \\ b \\ c \end{pmatrix}; \mu_{s,t}\Sigma\right) \quad \text{where} \quad \Sigma = \begin{pmatrix} a(1-a) & -ab & -ac \\ -ab & b(1-b) & -bc \\ -ac & -bc & c(1-c) \end{pmatrix}$$

Similarly for $s = 1$, we have

$$f_{t+1}^{\cdot,1} = \begin{pmatrix} f_{t+1}^{1,1} \\ f_{t+1}^{2,1} \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\mu_{1,t}\begin{pmatrix} 1-c \\ c \end{pmatrix}; \mu_{1,t}\Sigma_1\right) \quad \text{where} \quad \Sigma_1 = \begin{pmatrix} c(1-c) & -c(1-c) \\ -c(1-c) & c(1-c) \end{pmatrix}$$

and for $s = S$, we have

$$f_{t+1}^{\cdot,S} = \begin{pmatrix} f_{t+1}^{S-1,S} \\ f_{t+1}^{S,S} \end{pmatrix} \rightsquigarrow \mathcal{N}\left(\mu_{S,t}\begin{pmatrix} a \\ 1-a \end{pmatrix}; \mu_{S,t}\Sigma_S\right) \quad \text{where} \quad \Sigma_S = \begin{pmatrix} a(1-a) & -a(1-a) \\ -a(1-a) & a(1-a) \end{pmatrix}$$

It follows that we can rewrite the vector $f_{t+1}^{\cdot,s}$ as

$$\begin{aligned} f_{t+1}^{\cdot,1} &= \mu_{1,t}\begin{pmatrix} 1-c \\ c \end{pmatrix} + \sqrt{\mu_{1,t}}\epsilon_{t+1}^{\cdot,1} \\ f_{t+1}^{\cdot,s} &= \mu_{s,t}\begin{pmatrix} a \\ b \\ c \end{pmatrix} + \sqrt{\mu_{s,t}}\epsilon_{t+1}^{\cdot,s} \quad \text{for} \quad 1 < s < S \\ f_{t+1}^{\cdot,S} &= \mu_{S,t}\begin{pmatrix} a \\ 1-a \end{pmatrix} + \sqrt{\mu_{S,t}}\epsilon_{t+1}^{\cdot,S} \end{aligned}$$

where $\epsilon_{t+1}^{\cdot,1} \rightsquigarrow \mathcal{N}(0,\Sigma_1)$, $\epsilon_{t+1}^{\cdot,s} \rightsquigarrow \mathcal{N}(0,\Sigma)$ for $1 < s < S$, and, $\epsilon_{t+1}^{\cdot,S} \rightsquigarrow \mathcal{N}(0,\Sigma_S)$. Note that the $\epsilon_{t+1}^{\cdot,s}$ are then independent of the $\mu_{s,t}$. Let us introduce some notation that turns out to be useful:

$$I_s \equiv \begin{pmatrix} 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \\ \vdots & \vdots & \vdots \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad I_1 \equiv \begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \end{pmatrix} \quad \text{and} \quad I_S \equiv \begin{pmatrix} 0 & 0 \\ \vdots & \vdots \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{pmatrix}$$

where the s^{th} row of I_s is $(0,1,0)$. With this notation, it is easy to see that

$$\begin{aligned} \mu_t &= \sum_{s=1}^{S} I_s f_{t+1}^{\cdot,s} \\ &= \mu_{1,t} I_1 \begin{pmatrix} 1-c \\ c \end{pmatrix} + \sum_{s=2}^{S-1} \mu_{s,t} I_s \begin{pmatrix} a \\ b \\ c \end{pmatrix} + \mu_{S,t} I_S \begin{pmatrix} a \\ 1-a \end{pmatrix} + \sum_{s=1}^{S} I_s \sqrt{\mu_{s,t}} \epsilon_{t+1}^{\cdot,s} \end{aligned}$$

from which it follows that

$$\mu_t = P'\mu_t + \sum_{s=1}^{S} (I_s \epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}' e_s) \tag{30}$$

where P is the transition matrix of the idiosyncratic productivity process in Assumption 1, e_s is the s^{th} base vector, and, $\sqrt{\mu_t} = (\sqrt{\mu_{1,t}}, \ldots, \sqrt{\mu_{s,t}}, \ldots, \sqrt{\mu_{S,t}})'$.

Let us call the vector $\mu = \mathbb{E}[\mu_t]$, the unconditional expectation of the productivity distribution μ_t. From Equation 30 it is easy to show that μ satisfies $\mu = P'\mu$. Using a similar approach to the proof of Corollary 2 and the fact that $\sum_{s=1}^{S} \mu_s = N$, one can show that

$$\mu_s = \mathbb{E}[\mu_{s,t}] = N \frac{1-\varphi^{-\delta}}{\varphi^{-\delta}(1-(\varphi^S)^{-\delta})} (\varphi^s)^{-\delta}$$

To compute the unconditional variance-covariance matrix of μ_t, let us take the variance of Equation

30:

$$\begin{aligned}
\mathbb{V}ar\left[\mu_t\right] =& \mathbb{V}ar\left[P'\mu_t + \sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] \\
&= \mathbb{C}ov\left[P'\mu_t + \sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s); P'\mu_t + \sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] \\
&= \mathbb{C}ov\left[P'\mu_t; P'\mu_t\right] + \mathbb{C}ov\left[P'\mu_t; \sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] + \mathbb{C}ov\left[\sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s); P'\mu_t\right]\ldots \\
&\ldots + \mathbb{C}ov\left[\sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s); \sum_{s=1}^{S}(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] \\
&= P'\mathbb{V}ar\left[\mu_t\right]P + \sum_{s=1}^{S}P'\mathbb{C}ov\left[\mu_t; (I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] + \sum_{s=1}^{S}\left(P'\mathbb{C}ov\left[\mu_t; (I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right]\right)'\ldots \\
&\ldots + \sum_{s=1}^{S}\sum_{s'=1}^{S}\mathbb{C}ov\left[(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s); (I_{s'}\epsilon_{t+1}^{\cdot,s'})(\sqrt{\mu_t}'e_{s'})\right]
\end{aligned}$$

Note that $\mathbb{E}\left[(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] = I_s\mathbb{E}\left[(\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}')\right]e_s = I_s\mathbb{E}\left[(\epsilon_{t+1}^{\cdot,s})\right]\mathbb{E}\left[(\sqrt{\mu_t}')\right]e_s = 0$ since $\mathbb{E}\left[\epsilon_{t+1}^{\cdot,s}\right] = 0$, and, $\epsilon_{t+1}^{\cdot,s}$ and μ_t are independent. Let us look at the second and third term of the equation above:

$$\begin{aligned}
P'\mathbb{C}ov\left[\mu_t; (I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right] &= \mathbb{E}\left[(\mu_t - \mu)\left((I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\right)'\right] \\
&= \mathbb{E}\left[(\mu_t - \mu)(\sqrt{\mu_t}'e_s)'(I_s\epsilon_{t+1}^{\cdot,s})'\right] \\
&= \mathbb{E}\left[(\mu_t - \mu)(\sqrt{\mu_t}'e_s)'\right]\mathbb{E}\left[(I_s\epsilon_{t+1}^{\cdot,s})'\right] = 0
\end{aligned}$$

since $\mathbb{E}\left[\epsilon_{t+1}^{\cdot,s}\right] = 0$, and, $\epsilon_{t+1}^{\cdot,s}$ and μ_t are independent. Let us now look at the last term:

$$\begin{aligned}
\sum_{s=1}^{S}\sum_{s'=1}^{S}\mathbb{C}ov\left[(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s); (I_{s'}\epsilon_{t+1}^{\cdot,s'})(\sqrt{\mu_t}'e_{s'})\right] &= \sum_{s=1}^{S}\sum_{s'=1}^{S}\mathbb{E}\left[(I_s\epsilon_{t+1}^{\cdot,s})(\sqrt{\mu_t}'e_s)\left((I_{s'}\epsilon_{t+1}^{\cdot,s'})(\sqrt{\mu_t}'e_{s'})\right)'\right] \\
&= \sum_{s=1}^{S}\sum_{s'=1}^{S}\mathbb{E}\left[(I_s\epsilon_{t+1}^{\cdot,s})\sqrt{\mu_t}'e_s e_{s'}'\sqrt{\mu_t}(I_{s'}\epsilon_{t+1}^{\cdot,s'})'\right] \\
&= \sum_{s=1}^{S}\mathbb{E}\left[\mu_{s,t}\right]I_s\mathbb{E}\left[\epsilon_{t+1}^{\cdot,s}(\epsilon_{t+1}^{\cdot,s})'\right]I_s' \\
&= \mu_1 I_1\Sigma_1 I_1' + \sum_{s=2}^{S-1}\mu_s I_s\Sigma I_s' + \mu_S I_S\Sigma_S I_S'
\end{aligned}$$

where in the fourth line we use the fact that if $s \neq s'$ then $\sqrt{\mu_t}'e_s e_{s'}'\sqrt{\mu_t} = 0$ and if $s = s'$ then $\sqrt{\mu_t}'e_s e_{s'}'\sqrt{\mu_t} = \mu_{s,t}$. The variance-covariance matrix of μ_t is thus characterized by the following discrete Lyapunov equation:

$$\mathbb{V}ar\left[\mu_t\right] = P'\mathbb{V}ar\left[\mu_t\right]P + \mu_1 I_1\Sigma_1 I_1' + \sum_{s=2}^{S-1}\mu_s I_s\Sigma I_s' + \mu_S I_S\Sigma_S I_S' \tag{31}$$

The solution of the discrete Lyapunov Equation 31 is thus:

$$\mathbb{V}ar\left[\mu_t\right] = \sum_{k=0}^{\infty}(P')^k\left(\mu_1 I_1 \Sigma_1 I_1' + \sum_{s=2}^{S-1}\mu_s I_s \Sigma I_s' + \mu_S I_S \Sigma_S I_S'\right)P^k$$

note that $I_s\Sigma I_s' = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \Sigma & 0 \\ 0 & 0 & 0 \end{pmatrix}$, $I_1\Sigma_1 I_1' = \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix}$ and $I_S\Sigma_S I_S' = \begin{pmatrix} 0 & 0 \\ 0 & \Sigma_S \end{pmatrix}$. □

B.5 Proof of Theorem 2

In this appendix, we state and prove the more general Theorem 3 which extends the results of Theorem 2 to the entry and exit case. Formally, we show that the following theorem is true:

Theorem 3 *Assume 1, then*

(i) *The dynamic of aggregate productivity is given by*

$$A_{t+1} = \rho A_t + \rho E_t(\varphi) + O_t^A + \sigma_t \varepsilon_{t+1} \tag{32}$$

$$\sigma_t^2 = \varrho D_t + \varrho E_t(\varphi^2) + O_t^\sigma \tag{33}$$

where $\mathbb{E}[\varepsilon_{t+1}] = 0$ *and* $\mathbb{V}ar[\varepsilon_{t+1}] = 1$. *The persistence of the aggregate state is* $\rho = a\varphi^{\frac{-1}{1-\alpha}} + b + c\varphi^{\frac{1}{1-\alpha}}$. *The term* D_t *is given by* $D_t := \sum_{s=s^*(\mu_t)-1}^{S}\left((\varphi^s)^{\frac{1}{1-\alpha}}\right)^2\mu_{s,t}$ *and* $\varrho = a\varphi^{\frac{-2}{1-\alpha}} + b + c\varphi^{\frac{2}{1-\alpha}} - \rho^2$. *The terms* $E_t(\varphi)$ *and* $E_t(\varphi^2)$ *are defined using the* $E_t(x) = \sum_{s=s_t^*}^{S} x^s MG_s - x^{\frac{s_t^*-1}{1-\alpha}}\mu_{s_t^*-1,t}$ *for any* x. *The terms* O_t^A *and* O_t^σ *are a correction for the upper and lower reflecting barriers in the idiosyncratic state space definied in the proof. Furthermore, for a large number of firms the distribution of* ε_{t+1} *can be approximate by a standard normal distribution.*

(ii) *Aggregate output (in percentage deviation from its steady-state value) has the following law of motion:*

$$\widehat{Y}_{t+1} = \rho\widehat{Y}_t + \kappa\widehat{O_t^A} + \psi\frac{\sigma_t}{A}\epsilon_{t+1} \tag{34}$$

$\widehat{O_t^A}$ *is the percentage deviation from steady-state of* O_t^A, κ *and* ψ *are constants defined below and* A *is the steady-state value of the aggregate productivity* A_t.

Proof: Aggregate productivity

Note first that

$$A_{t+1} = \sum_{i=1}^{N_{t+1}}\varphi^{\frac{s_{t+1,i}}{1-\alpha}} = \sum_{s=1}^{S}\varphi^{\frac{s}{1-\alpha}}\mu_{s,t+1}$$

where $\mu_{s,t+1}$, the number of firms in productivity bin s at time $t+1$, is stochastic as shown in Theorem 1. Using the proof of this theorem for $S > s > s^*(\mu_t)$ and under Assumption 1, we have:

$$\mu_{s,t+1} = f_{t+1}^{s,s-1} + f_{t+1}^{s,s} + f_{t+1}^{s,s+1} + g_{t+1}^{s,s-1} + g_{t+1}^{s,s} + g_{t+1}^{s,s+1}$$

where $f_{k,t+1}^{s',s}$ is the number of firms in state s' at $t+1$ that were in state s at time t and $g_{k,t+1}^{s',s}$ is the number of entrants in state s' at $t+1$ that received a signal s at time t. Given Assumption 1 the 3×1 vector $f_{k,t+1}^{:,s} = (f_{t+1}^{s-1,s}, f_{t+1}^{s,s}, f_{t+1}^{s+1,s})'$ follows a multinomial distribution with number of trials $\mu_{s,t+1}$ and event probabilities $(a,b,c)'$. Similarly, the 3×1 vector $g_{k,t+1}^{:,s} = (g_{t+1}^{s-1,s}, g_{t+1}^{s,s}, g_{t+1}^{s+1,s})'$ follows a multinomial distribution with number of trials MG_s and event probabilities $(a, b, c)'$. In other words, for $S > s \geq s^*(\mu_t)$:

$$f_{t+1}^{\cdot,s} = \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{s,t}, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right) \qquad \text{and} \qquad g_{t+1}^{\cdot,s} = \begin{pmatrix} g_{t+1}^{s-1,s} \\ g_{t+1}^{s,s} \\ g_{t+1}^{s+1,s} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(MG_s, \begin{pmatrix} a \\ b \\ c \end{pmatrix}\right)$$

Furthermore, we also have:

$$\begin{aligned}
\mu_{s^*(\mu_t)-1,t+1} &= f_{t+1}^{s^*(\mu_t)-1,s^*(\mu_t)} + g_{t+1}^{s^*(\mu_t)-1,s^*(\mu_t)} \\
\mu_{s^*(\mu_t),t+1} &= f_{t+1}^{s^*(\mu_t),s^*(\mu_t)} + f_{t+1}^{s^*(\mu_t),s^*(\mu_t)+1} + g_{t+1}^{s^*(\mu_t),s^*(\mu_t)} + g_{t+1}^{s^*(\mu_t),s^*(\mu_t)+1} \\
\mu_{S,t+1} &= f_{t+1}^{S,S-1} + f_{t+1}^{S,S} + g_{t+1}^{S,S-1} + g_{t+1}^{S,S}
\end{aligned}$$

Note that we have

$$f_{t+1}^{\cdot,S} = \begin{pmatrix} f_{t+1}^{S-1,S} \\ f_{t+1}^{S,S} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{S,t}, \begin{pmatrix} a \\ b+c \end{pmatrix}\right) \qquad \text{and} \qquad g_{t+1}^{\cdot,S} = \begin{pmatrix} g_{t+1}^{S-1,S} \\ g_{t+1}^{S,S} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(MG_S, \begin{pmatrix} a \\ b+c \end{pmatrix}\right)$$

Having shown these preliminary results, let us consider: [38]

$$\begin{aligned}
A_{t+1} =& \sum_{s=1}^{S} (\varphi^{\frac{1}{1-\alpha}})^s \mu_{s,t+1} = (\varphi^{\frac{1}{1-\alpha}})^{s_t^*-1} \mu_{s_t^*-1,t+1} + (\varphi^{\frac{1}{1-\alpha}})^{s_t^*} \mu_{s_t^*,t+1} + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \mu_{s,t+1} + (\varphi^{\frac{1}{1-\alpha}})^S \mu_{S,t+1} \\
=& (\varphi^{\frac{1}{1-\alpha}})^{s_t^*-1} \left(f_{t+1}^{s_t^*-1,s_t^*} + g_{t+1}^{s_t^*-1,s_t^*}\right) + (\varphi^{\frac{1}{1-\alpha}})^{s_t^*} \left(f_{t+1}^{s_t^*,s_t^*} + f_{t+1}^{s_t^*,s_t^*+1} + g_{t+1}^{s_t^*,s_t^*} + g_{t+1}^{s_t^*,s_t^*+1}\right) \\
& \ldots + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s-1} + f_{t+1}^{s,s} + f_{t+1}^{s,s+1} + g_{t+1}^{s,s-1} + g_{t+1}^{s,s} + g_{t+1}^{s,s+1}\right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(f_{t+1}^{S,S-1} + f_{t+1}^{S,S} + g_{t+1}^{S,S-1} + g_{t+1}^{S,S}\right) \\
=& (\varphi^{\frac{1}{1-\alpha}})^{s_t^*-1} \left(f_{t+1}^{s_t^*-1,s_t^*} + (\varphi^{\frac{1}{1-\alpha}}) f_{t+1}^{s_t^*,s_t^*} + g_{t+1}^{s_t^*-1,s_t^*} + (\varphi^{\frac{1}{1-\alpha}}) g_{t+1}^{s_t^*,s_t^*}\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^{s_t^*} \left(f_{t+1}^{s_t^*,s_t^*+1} + g_{t+1}^{s_t^*,s_t^*+1}\right) + \\
& \ldots + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s-1} + g_{t+1}^{s,s-1}\right) + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s} + g_{t+1}^{s,s}\right) + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s+1} + g_{t+1}^{s,s+1}\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^S \left(f_{t+1}^{S,S-1} + g_{t+1}^{S,S-1}\right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(f_{t+1}^{S,S} + g_{t+1}^{S,S}\right) \\
=& (\varphi^{\frac{1}{1-\alpha}})^{s_t^*-1} \left(f_{t+1}^{s_t^*-1,s_t^*} + (\varphi^{\frac{1}{1-\alpha}}) f_{t+1}^{s_t^*,s_t^*} + g_{t+1}^{s_t^*-1,s_t^*} + (\varphi^{\frac{1}{1-\alpha}}) g_{t+1}^{s_t^*,s_t^*}\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^{s_t^*} \left(f_{t+1}^{s_t^*,s_t^*+1} + g_{t+1}^{s_t^*,s_t^*+1}\right) + \\
& \ldots + \varphi^{\frac{1}{1-\alpha}} \sum_{s=s_t^*}^{S-2} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s+1,s} + g_{t+1}^{s+1,s}\right) + \sum_{s=s_t^*+1}^{S-1} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s} + g_{t+1}^{s,s}\right) + (\varphi^{\frac{1}{1-\alpha}})^{-1} \sum_{s=s_t^*+2}^{S} (\varphi^{\frac{1}{1-\alpha}})^{s-1} \left(f_{t+1}^{s-1,s} + g_{t+1}^{s-1,s}\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^S \left(f_{t+1}^{S,S-1} + g_{t+1}^{S,S-1}\right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(f_{t+1}^{S,S} + g_{t+1}^{S,S}\right) \\
=& (\varphi^{\frac{1}{1-\alpha}})^{s_t^*} \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{s_t^*-1,s_t^*} + g_{t+1}^{s_t^*-1,s_t^*}\right) + f_{t+1}^{s_t^*,s_t^*} + g_{t+1}^{s_t^*,s_t^*} + \varphi^{\frac{1}{1-\alpha}} \left(f_{t+1}^{s_t^*+1,s_t^*} + g_{t+1}^{s_t^*+1,s_t^*}\right)\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^{s_t^*+1} \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{s_t^*,s_t^*+1} + g_{t+1}^{s_t^*,s_t^*+1}\right) + f_{t+1}^{s_t^*+1,s_t^*+1} + g_{t+1}^{s_t^*+1,s_t^*+1} + \varphi^{\frac{1}{1-\alpha}} \left(f_{t+1}^{s_t^*+2,s_t^*+1} + g_{t+1}^{s_t^*+2,s_t^*+1}\right)\right) + \\
& \ldots + \varphi^{\frac{1}{1-\alpha}} \sum_{s=s_t^*+2}^{S-2} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s+1,s} + g_{t+1}^{s+1,s}\right) + \sum_{s=s_t^*+2}^{S-2} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s,s} + g_{t+1}^{s,s}\right) + \varphi^{\frac{-1}{1-\alpha}} \sum_{s=s_t^*+2}^{S-2} (\varphi^{\frac{1}{1-\alpha}})^s \left(f_{t+1}^{s-1,s} + g_{t+1}^{s-1,s}\right) + \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^{S-1} \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{S-2,S-1} + g_{t+1}^{S-2,S-1}\right) + f_{t+1}^{S-1,S-1} + g_{t+1}^{S-1,S-1} + \varphi^{\frac{1}{1-\alpha}} \left(f_{t+1}^{S,S-1} + g_{t+1}^{S,S-1}\right)\right) \\
& \ldots + (\varphi^{\frac{1}{1-\alpha}})^S \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{S-1,S} + g_{t+1}^{S-1,S}\right) + f_{t+1}^{S,S} + g_{t+1}^{S,S}\right) \\
=& \sum_{s=s_t^*}^{S-1} \varphi^{\frac{s}{1-\alpha}} \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{s-1,s} + g_{t+1}^{s-1,s}\right) + f_{t+1}^{s,s} + g_{t+1}^{s,s} + \varphi^{\frac{1}{1-\alpha}} \left(f_{t+1}^{s+1,s} + g_{t+1}^{s+1,s}\right)\right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(\varphi^{\frac{-1}{1-\alpha}} \left(f_{t+1}^{S-1,S} + g_{t+1}^{S-1,S}\right) + f_{t+1}^{S,S} + g_{t+1}^{S,S}\right) \\
=& \sum_{s=s_t^*}^{S-1} \varphi^{\frac{s}{1-\alpha}} \left(\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix} + \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} g_{t+1}^{s-1,s} \\ g_{t+1}^{s,s} \\ g_{t+1}^{s+1,s} \end{pmatrix}\right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{t+1}^{S-1,S} \\ f_{t+1}^{S,S} \end{pmatrix} + \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} g_{t+1}^{S-1,S} \\ g_{t+1}^{S,S} \end{pmatrix}\right)
\end{aligned}$$

[38]note that we use the notation s_t^* instead of $s^*(\mu_t)$ to keep the notation parsimonious)

It is easy to see that for $s < S$,

$$\mathbb{E}\left[\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix}\right] = \mu_{s,t} \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} a \\ b \\ c \end{pmatrix} = \rho\mu_{s,t} \quad \text{and} \quad \mathbb{V}ar\left[\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix}\right] = \mu_{s,t} \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \Sigma \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix} = \varrho\mu_{s,t}$$

with

$$\Sigma = \begin{pmatrix} a(1-a) & -ab & -ac \\ -ab & b(1-b) & -bc \\ -ac & -bc & c(1-c) \end{pmatrix}$$

from which it follows that

$$\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix} = \rho\mu_{s,t} + \sqrt{\varrho\mu_{s,t}}\varepsilon_{s,t+1}^{f}$$

where $\varepsilon_{s,t+1}^{f}$ is a mean-zero, unit variance random variable, independent across time and state s (and independent of $\mu_{s,t}$). Furthermore, using the approximation of a multinomial by a multivariate distribution we can see that $\varepsilon_{s,t} \rightsquigarrow \mathcal{N}(0,1)$ for large $\mu_{s,t}$ (see p377 example 12.7 in Severini (2005)). Similarly,

$$\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} g_{t+1}^{s-1,s} \\ g_{t+1}^{s,s} \\ g_{t+1}^{s+1,s} \end{pmatrix} = \rho MG_s + \sqrt{\varrho MG_s}\varepsilon_{s,t+1}^{g}$$

where $\varepsilon_{s,t+1}^{g}$ is a mean-zero, unit variance random variable independent across time and state s. This again can be approximated by a standard normal distribution $\mathcal{N}(0,1)$. Using the same reasoning, we have

$$\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{t+1}^{S-1,S} \\ f_{t+1}^{S,S} \end{pmatrix} = \rho_S\mu_{S,t} + \sqrt{\varrho_S\mu_{S,t}}\varepsilon_{S,t+1}^{f} \quad \text{and} \quad \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} g_{t+1}^{S-1,S} \\ g_{t+1}^{S,S} \end{pmatrix} = \rho_S MG_S + \sqrt{\varrho_S MG_S}\varepsilon_{S,t+1}^{g}$$

where $\varepsilon_{S,t+1}^{f}$ and $\varepsilon_{S,t+1}^{g}$ is a mean-zero, unit variance random variable independent across time and state for $s \neq S$. This can be approximated by a standard normal distribution $\mathcal{N}(0,1)$. Finally,

$$\rho_S = \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} a \\ b+c \end{pmatrix} \quad \text{and} \quad \varrho_S = \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} a(1-a) & -a(1-a) \\ -a(1-a) & a(1-a) \end{pmatrix} \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}$$

Let us these results to compute A_{t+1}

$$\begin{aligned}
A_{t+1} &= \sum_{s=s_t^*}^{S-1} \varphi^{\frac{s}{1-\alpha}} \left(\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} f_{t+1}^{s-1,s} \\ f_{t+1}^{s,s} \\ f_{t+1}^{s+1,s} \end{pmatrix} + \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \\ \varphi^{\frac{1}{1-\alpha}} \end{pmatrix}' \begin{pmatrix} g_{t+1}^{s-1,s} \\ g_{t+1}^{s,s} \\ g_{t+1}^{s+1,s} \end{pmatrix} \right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(\begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} f_{t+1}^{S-1,S} \\ f_{t+1}^{S,S} \end{pmatrix} + \begin{pmatrix} \varphi^{\frac{-1}{1-\alpha}} \\ 1 \end{pmatrix}' \begin{pmatrix} g_{t+1}^{S-1,S} \\ g_{t+1}^{S,S} \end{pmatrix} \right) \\
&= \sum_{s=s_t^*}^{S-1} \varphi^{\frac{s}{1-\alpha}} \left(\rho\mu_{s,t} + \sqrt{\varrho\mu_{s,t}}\varepsilon_{s,t+1}^{f} + \rho MG_s + \sqrt{\varrho MG_s}\varepsilon_{s,t+1}^{g} \right) + (\varphi^{\frac{1}{1-\alpha}})^S \left(\rho_S\mu_{S,t} + \sqrt{\varrho_S\mu_{S,t}}\varepsilon_{S,t+1}^{f} + \rho_S MG_S + \sqrt{\varrho_S MG_S}\varepsilon_{S,t+1}^{g} \right) \\
&= \rho \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \mu_{s,t} + \rho \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} MG_s + \sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \left(\sqrt{\mu_{s,t}}\varepsilon_{s,t+1}^{f} + \sqrt{MG_s}\varepsilon_{s,t+1}^{g} \right) + \dots \\
&\quad \dots + (\varphi^{\frac{1}{1-\alpha}})^S \left((\rho_S - \rho)\mu_{S,t} + (\rho_S - \rho)MG_S + \left(\sqrt{\varrho_S\mu_{S,t}} - \sqrt{\varrho\mu_{S,t}} \right) \varepsilon_{S,t+1}^{f} + \left(\sqrt{\varrho_S MG_S} - \sqrt{\varrho MG_S} \right) \varepsilon_{S,t+1}^{g} \right) \\
&= \rho \sum_{s=s_t^*-1}^{S} \varphi^{\frac{s}{1-\alpha}} \mu_{s,t} + \rho \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} MG_s - \rho\varphi^{\frac{s_t^*-1}{1-\alpha}} \mu_{s_t^*-1,t} + \sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \left(\sqrt{\mu_{s,t}}\varepsilon_{s,t+1}^{f} + \sqrt{MG_s}\varepsilon_{s,t+1}^{g} \right) + \dots \\
&\quad \dots + (\varphi^{\frac{1}{1-\alpha}})^S \left((\rho_S - \rho)\mu_{S,t} + (\rho_S - \rho)MG_S + \left(\sqrt{\varrho_S\mu_{S,t}} - \sqrt{\varrho\mu_{S,t}} \right) \varepsilon_{S,t+1}^{f} + \left(\sqrt{\varrho_S MG_S} - \sqrt{\varrho MG_S} \right) \varepsilon_{S,t+1}^{g} \right) \\
&= \rho \sum_{s=s_t^*-1}^{S} \varphi^{\frac{s}{1-\alpha}} \mu_{s,t} + \rho \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} MG_s - \rho\varphi^{\frac{s_t^*-1}{1-\alpha}} \mu_{s_t^*-1,t} + (\varphi^{\frac{1}{1-\alpha}})^S \left((\rho_S - \rho)\mu_{S,t} + (\rho_S - \rho)MG_S \right) + \dots \\
&\quad \dots + \sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \sqrt{\mu_{s,t}}\varepsilon_{s,t+1}^{f} + \sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \sqrt{MG_s}\varepsilon_{s,t+1}^{g} + (\varphi^{\frac{1}{1-\alpha}})^S \left(\left(\sqrt{\varrho_S\mu_{S,t}} - \sqrt{\varrho\mu_{S,t}} \right) \varepsilon_{S,t+1}^{f} + \left(\sqrt{\varrho_S MG_S} - \sqrt{\varrho MG_S} \right) \varepsilon_{S,t+1}^{g} \right)
\end{aligned}$$

Note that, by definition, $A_t = \sum_{s=s_t^*-1}^{S} \varphi^{\frac{s}{1-\alpha}} \mu_{s,t}$ and $E_t(\varphi) = \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} MG_s - \varphi^{\frac{s_t^*-1}{1-\alpha}} \mu_{s_t^*-1,t}$. We define $O_t^A \equiv (\varphi^{\frac{1}{1-\alpha}})^S ((\rho_S - \rho)\mu_{S,t} + (\rho_S - \rho)MG_S)$. Furthermore,

$$\begin{aligned}
\mathbb{V}ar_t\left[A_{t+1}\right] = \sigma_t^2 =& \mathbb{V}ar\left[\sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \sqrt{\mu_{s,t}} \varepsilon_{s,t}^f + \sqrt{\varrho} \sum_{s=s_t^*}^{S} \varphi^{\frac{s}{1-\alpha}} \sqrt{MG_s} \varepsilon_{s,t+1}^g + (\varphi^{\frac{1}{1-\alpha}})^S \left(\left(\sqrt{\varrho_S \mu_{S,t+1}} - \sqrt{\varrho \mu_{S,t}}\right) \varepsilon_{S,t+1}^f + \left(\sqrt{\varrho_S MG_S} - \sqrt{\varrho MG_S}\right) \varepsilon_{S,t+1}^g\right)\right] \\
=& \varrho \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} \mu_{s,t} + \varrho \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} MG_s + \varphi^{\frac{2S}{1-\alpha}} \left((\sqrt{\varrho_S} - \sqrt{\varrho})^2 \mu_{S,t} + (\sqrt{\varrho_S} - \sqrt{\varrho})^2 MG_S\right) \\
=& \varrho \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} \mu_{s,t} + \varrho \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} MG_s + \varphi^{\frac{2S}{1-\alpha}} \left((\sqrt{\varrho_S} - \sqrt{\varrho})^2 \mu_{S,t} + (\sqrt{\varrho_S} - \sqrt{\varrho})^2 MG_S\right) \\
=& \varrho \sum_{s=s_t^*-1}^{S} \varphi^{\frac{2s}{1-\alpha}} \mu_{s,t} + \varrho \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} MG_s - \varrho \varphi^{\frac{2(s_t^*-1)}{1-\alpha}} \mu_{s_t^*,t} + \varphi^{\frac{2S}{1-\alpha}} \left((\sqrt{\varrho_S} - \sqrt{\varrho})^2 \mu_{S,t} + (\sqrt{\varrho_S} - \sqrt{\varrho})^2 MG_S\right)
\end{aligned}$$

Note that $D_t = \sum_{s=s_t^*-1}^{S} \varphi^{\frac{2s}{1-\alpha}} \mu_{s,t}$ while $E_t(\varphi^2) = \sum_{s=s_t^*}^{S} \varphi^{\frac{2s}{1-\alpha}} MG_s - \varphi^{\frac{2(s_t^*-1)}{1-\alpha}} \mu_{s_t^*,t}$ and we define $O_t^\sigma \equiv \varphi^{\frac{2S}{1-\alpha}} \left(\left(\sqrt{\varrho_S} - \sqrt{\varrho}\right)^2 \mu_{S,t} + \left(\sqrt{\varrho_S} - \sqrt{\varrho}\right)^2 MG_S\right)$. It follows that

$$A_{t+1} = \rho A_t + \rho E_t(\varphi) + O_t^A + \sigma_t \varepsilon_{t+1} \quad \text{where} \quad \sigma_t = \varrho D_t + \varrho E_t(\varphi^2) + O_t^\sigma$$

with ε_{t+1} a mean zero and unit variance random variable. When using the approximation of a multinomial by a multivariate normal distribution, it is easy to show that ε_{t+1} follow a standard normal distribution. The above proof applies to the no entry-exit case with little changes using the fact that

$$f_{t+1}^{\cdot,1} = \begin{pmatrix} f_{t+1}^{1,1} \\ f_{t+1}^{2,1} \end{pmatrix} \rightsquigarrow \mathcal{M}ulti\left(\mu_{1,t}, \left(\begin{smallmatrix} a+b \\ c \end{smallmatrix}\right)\right)$$

This completes the proof of the law of motion of aggregate productivity A_t. □

Proof: Aggregate Output

To prove the law of motion of aggregate output (in percentage deviation from its steady-state value), we first solve for aggregate output, Y_t, as a function of the univariate state variable A_t analytically. We then study their first order relationship. The next step is then to take the first-order approximation of the equation describing the dynamics of A_t. Finally, we find the implied first-order dynamics of Y_t.

Let us first compute aggregate output Y_t as a function of A_t only:

$$Y_t = \sum_{i=1}^{N_t} y_t^i = \sum_{s=1}^{S} \mu_{s,t} (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} = \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} A_t$$

Recall that $w_t = \left(\alpha^{\frac{1}{1-\alpha}} \frac{A_t}{LM}\right)^{\frac{1-\alpha}{\gamma(1-\alpha)+1}}$. Substituting the expression of the wage in the latter equation yields $Y_t = \alpha^{\frac{\alpha\gamma}{\gamma(1-\alpha)+1}} \left(\frac{1}{L(M)}\right)^{\frac{-\alpha}{\gamma(1-\alpha)+1}} (A_t)^{1-\frac{\alpha}{\gamma(1-\alpha)+1}}$. This last equality, taken at the first order, implies that:

$$\widehat{Y}_t = \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right) \widehat{A}_t \tag{35}$$

where $\widehat{X}_t$ of a variable X_t is the percentage deviation from its steady-state value X: $\widehat{X}_t = (X_t - X)/X$. Let us define $\psi \equiv \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)$.

We then take the percentage deviation from steady-state of Equation 32:

$$\begin{aligned}
A_{t+1} =& \rho A_t + \rho E_t + O_t^A + \sigma_t \varepsilon_{t+1} \\
A =& \rho A + \rho E + O^A \\
A_{t+1} - A =& \rho(A_t - A) + \rho(E_t - E) + (O_t^A - O) + \sigma_t \varepsilon_{t+1} \\
\frac{A_{t+1} - A}{A} =& \rho \frac{A_t - A}{A} + \rho \frac{E}{T} \frac{E_t - E}{E} + \frac{O^A}{A} \frac{O_t^A - O}{O} + \frac{\sigma_t}{A} \varepsilon_{t+1}
\end{aligned}$$

$$\widehat{A_{t+1}} = \rho\widehat{A_t} + \rho\frac{E}{A}\widehat{E_t} + \frac{O^A}{A}\widehat{O_t^A} + \frac{\sigma_t}{A}\varepsilon_{t+1}$$
$$\widehat{Y}_{t+1} = \rho\widehat{Y}_t + \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\rho\frac{E}{A}\widehat{E}_t + \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\frac{O^A}{A}\widehat{O_t^A} + \left(1 - \frac{\alpha}{\gamma(1-\alpha)+1}\right)\frac{\sigma_t}{A}\varepsilon_{t+1}$$

where the second line is Equation 32 at the steady-state; in the third line we subtract the second from the first line; in the fourth line we divide both sides by the steady-state value of A and in the last line we use Equation 35. □

B.6 Proof of Proposition 3: Aggregate Persistence

In this appendix, we prove Proposition 3 regarding the comparative statics results for aggregate persistence, ρ. We first express ρ as a function of b, a measure of micro-level persistence, and of δ, the tail of the productivity stationary distribution.

First, note that from definition $\delta = \frac{\log(a/c)}{\log\varphi}$, it follows that $c = a\varphi^{-\delta}$. Secondly, from the fact that $b = 1 - a - c = 1 - a(1 + \varphi^{-\delta})$ we have that $a = \frac{1-b}{1+\varphi^{-\delta}}$. From Theorem 2, aggregate persistence is $\rho = a\varphi^{\frac{-1}{1-\alpha}} + b + c\varphi^{\frac{1}{1-\alpha}}$. In this last equation, let us substitute c and a using $c = a\varphi^{-\delta}$ and $a = \frac{1-b}{1+\varphi^{-\delta}}$:

$$\rho = \frac{1-b}{1+\varphi^{-\delta}}\varphi^{\frac{-1}{1-\alpha}} + b + \varphi^{-\delta}\varphi^{\frac{1}{1-\alpha}}\frac{1-b}{1+\varphi^{-\delta}}$$
$$\rho = \frac{1-b}{1+\varphi^{-\delta}}\left[\varphi^{\frac{-1}{1-\alpha}} - \varphi^{-\delta} + \varphi^{-\delta}\varphi^{\frac{1}{1-\alpha}} - 1\right] + 1$$

First, it is clear that if $\delta = \frac{1}{1-\alpha}$, then it follows that $\rho = 1$. This is exactly (iii) of the Proposition 3.

Second, from the expression of ρ, it is clear that $\frac{d\rho}{db} > 0$ if and only if $g(\delta) = \varphi^{\frac{-1}{1-\alpha}} - \varphi^{-\delta} + \varphi^{-\delta}\varphi^{\frac{1}{1-\alpha}} - 1 < 0$. Note that $g(\frac{1}{1-\alpha}) = 0$ and $g(\delta) \underset{\delta\to\infty}{\longrightarrow} \varphi^{\frac{-1}{1-\alpha}} - 1 < 0$ since $\varphi > 1$. The derivative of g is $g'(\delta) = -(-\log\varphi)\varphi^{-\delta} + (-\log\varphi)\varphi^{-\delta+\frac{1}{1-\alpha}} < 0$. It follows that for $\delta > \frac{1}{1-\alpha}$, then $g(\delta) < 0$ and thus $\frac{d\rho}{db} > 0$. We have just shown (i).

Finally to show (ii), let us rewrite $\rho = -\frac{(b-1)g(\delta)}{1+\varphi^{-\delta}} + 1$. We have shown that for $g(\delta)$ is decreasing in δ, since $b < 1$ it is clear that $(b-1)g(\delta)$ is increasing in δ. Note that $\frac{1}{1+\varphi^{-\delta}}$ is also increasing in δ. It follows that $\frac{(b-1)g(\delta)}{1+\varphi^{-\delta}}$ is increasing in δ which then implies that ρ is decreasing in δ, which is the statement in (ii).

□

B.7 Intermediate result: the link between the number of incumbents N and the number of potential entrants M

In this appendix, we are interested in the relationship between the number of incumbents N, the number of potentials entrants M, and the value of their ratio when N goes to infinity. We show that as N goes to infinity, the ratio M/N goes to a constant. This means that taking the endogenous variable N or the exogenous parameter M to infinity is strictly equivalent.

The number of firms is simply the sum of the number of firms in each bin:

$$
\begin{aligned}
N =& \sum_{s=1}^{S} \mu_s = \mu_{s^*-1} + \sum_{s=s^*}^{S} \mu_s \\
=& a\left(MK_eC_1 + MK_eC_2(\varphi^{s^*})^{-\delta_e} + MK_eC_3 + MK_e(\varphi^{s^*})^{-\delta_e}\right) + MK_eC_3 \sum_{s=s^*}^{S} 1 \\
&+ MK_eC_1(\varphi^{s^*})^{\delta} \sum_{s=s^*}^{S} (\varphi^s)^{-\delta} + MK_eC_2 \sum_{s=s^*}^{S} (\varphi^s)^{-\delta_e} \\
=& a\left(MK_eC_1 + MK_eC_2(\varphi^{s^*})^{-\delta_e} + MK_eC_3 + MK_e(\varphi^{s^*})^{-\delta_e}\right) + MK_eC_3(S - s^* + 1) \\
&+ MK_eC_1(\varphi^{s^*})^{\delta} \frac{(\varphi^{-\delta})^{s^*} - (\varphi^{-\delta})^{S}}{1-\varphi^{-\delta}} + MK_eC_2 \frac{(\varphi^{-\delta_e})^{s^*} - (\varphi^{-\delta_e})^{S}}{1-\varphi^{-\delta_e}}
\end{aligned}
$$

thus, by dividing both side by M, we have

$$
\frac{N}{M} = a\left(K_eC_1 + K_e(C_2+1)(\varphi^{s^*})^{-\delta_e} + K_eC_3\right) + K_eC_3(S-s^*+1) + K_eC_1(\varphi^{s^*})^{\delta} \frac{(\varphi^{-\delta})^{s^*} - (\varphi^{-\delta})^{S}}{1-\varphi^{-\delta}} + K_eC_2 \frac{(\varphi^{-\delta_e})^{s^*} - (\varphi^{-\delta_e})^{S}}{1-\varphi^{-\delta_e}}
$$

Let us note that under assumption 2

$$
\left(\varphi^{-\delta}\right)^{S} = \left(\varphi^{S}\right)^{-\delta} = \left(ZN^{1/\delta}\right)^{-\delta} = Z^{-\delta}N^{-1} \underset{N\to\infty}{\longrightarrow} 0
$$

and that, since $S = \frac{1}{\log\varphi}(\log Z + \frac{1}{\delta}\log N)$, we have

$$
SC_3 = \frac{1}{\log\varphi}(\log Z + \frac{1}{\delta}\log N)\frac{-\varphi^{-\delta_e}Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)}N^{-\delta_e/\delta} \underset{N\to\infty}{\longrightarrow} 0
$$

Thus, we have that

$$
\begin{aligned}
\frac{N}{M} \underset{N\to\infty}{\longrightarrow} (E^{\infty})^{-1} :=& a\left((\varphi^{\delta_e}-1)C_1^{\infty} + (\varphi^{\delta_e}-1)(C_2+1)(\varphi^{\overline{s^*}})^{-\delta_e}\right) \\
&+ (\varphi^{\delta_e}-1)C_1^{\infty}\frac{1}{1-\varphi^{-\delta}} + (\varphi^{\delta_e}-1)C_2\frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{1-\varphi^{-\delta_e}}
\end{aligned}
$$

where E^{∞} is the ratio of the number of potential entrants M and the number of incumbents, when there is an infinite number of incumbents. The last equation shows that M and N are equivalent when the number of incumbents is large. Thus, taking N to infinity is the same as taking M to infinity i.e $M \underset{N\to\infty}{\sim} E^{\infty}N$.

B.8 Proof of Propositions 4 and 5: Aggregate Volatility

In this appendix, we prove Proposition 5 describing how aggregate volatility decays with the number of firms N. This proof nests the proof of Proposition 4.

To prove this proposition, we study the asymptotic behavior of A, D and deduce the one for D/A^2, again when the number of firms N goes to infinity. We complete the proof by studying the behavior of the remaining terms $E(\varphi^2)$ and O^{σ} and $E(\varphi^2)/A^2$ and O^{σ}/A^2.

At this stage is important to note that, under Assumption 2, when N (and thus S) goes to infinity, the limit of the wage and the threshold are $\overline{w}$ and $\overline{s^*}$ which satisfies the system of equations given by the equation in Lemma 2 and in Proposition 1.

Step 0: Limit of the stationary distribution when the number of firms goes to infinity

The second step of the proof below will consist of finding the limit of constants K_e, C_1, C_2 and C_3 as the number of firms N goes to infinity. After finding these limits, we take the limit of Equation 25 in the previous lemma.

Here we first describe the asymptotic behavior of K_e. Recall that the entrant distribution sums to one.[39]

$$1 = \sum_{s=1}^{S} G_s = K_e \sum_{s=1}^{S} (\varphi^s)^{-\delta_e} = K_e \sum_{s=1}^{S} \left(\varphi^{-\delta_e}\right)^s = K_e \frac{\varphi^{-\delta_e} - \left(\varphi^{-\delta_e}\right)^{S+1}}{1 - \varphi^{-\delta_e}}$$

Rearranging terms, it follows that

$$K_e = \frac{1 - \varphi^{-\delta_e}}{\varphi^{-\delta_e} - (\varphi^{-\delta_e})^{S+1}}$$

Under Assumption 2 and since $\delta_e, \delta > 0$ we have

$$\left(\varphi^{-\delta_e}\right)^S = \left(\varphi^S\right)^{-\delta_e} = \left(ZN^{1/\delta}\right)^{-\delta_e} = Z^{-\delta_e} N^{-\delta_e/\delta} \underset{N\to\infty}{\longrightarrow} 0$$

by applying these results to the expression for K_e, it follows that $K_e \underset{N\to\infty}{\longrightarrow} \varphi^{\delta_e} - 1$.

Let us now focus on the asymptotic behavior of C_3, C_2 and C_1. From Lemma 1, we have

$$C_3 = \frac{-(\varphi^{-\delta_e})^{S+1}}{(1-\varphi^{-\delta_e})(a-c)} = \frac{-\varphi^{-\delta_e}(\varphi^S)^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} = \frac{-\varphi^{-\delta_e} Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} N^{-\delta_e/\delta} \underset{N\to\infty}{\longrightarrow} 0$$

We also have that

$$C_2 := \frac{\left(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c\right)}{\left(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c) + c\right)}$$

which is independent of S and thus of N.

Finally, we have

$$\begin{aligned} C_1 =& \frac{c\left(a(\varphi^{-\delta_e})^{S+2} - a(\varphi^{-\delta_e})^{s^*} - c(\varphi^{-\delta_e})^{S+3} + c(\varphi^{-\delta_e})^{s^*}\right)}{a(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e} - c)} \\ \underset{N\to\infty}{\longrightarrow}& \frac{c\left(-a(\varphi^{-\delta_e})^{\overline{s^*}} + c(\varphi^{-\delta_e})^{\overline{s^*}}\right)}{a(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e} - c)} = \frac{c}{a} \frac{-(a-c)(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(a-c)(a\varphi^{-\delta_e} - c)} \end{aligned}$$

and therefore

$$C_1 \underset{N\to\infty}{\longrightarrow} C_1^\infty := \frac{c}{a} \frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(c - a\varphi^{-\delta_e})}$$

We have just found the limit of K_e, C_1, C_2 and C_3 when N goes to infinity. We then apply these results to the stationary distribution by taking N to infinity. According to Lemma 1, we have for $s^* \leq s \leq S$:

$$\frac{\mu_s}{M} = K_e C_1 \left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + K_e C_2 (\varphi^s)^{-\delta_e} + K_e C_3$$

Under assumption 2, we have just shown that when the number of firms, N, goes to infinity, the stationary distribution is given by:

$$\frac{\mu_s}{M} = \left(\varphi^{\delta_e} - 1\right) \frac{c}{a} \frac{(\varphi^{-\delta_e})^{\overline{s^*}}}{(1-\varphi^{-\delta_e})(c - a\varphi^{-\delta_e})} \left(\frac{\varphi^s}{\varphi^{\overline{s^*}}}\right)^{-\delta} + \left(\varphi^{\delta_e} - 1\right) \frac{\left(a(\varphi^{-\delta_e})^2 + b\varphi^{-\delta_e} + c\right)}{\left(a(\varphi^{-\delta_e})^2 - \varphi^{-\delta_e}(a+c) + c\right)} (\varphi^s)^{-\delta_e}$$

Step 1: How A evolves when the number of incumbents converges to infinity

[39] Recall that we assume that G sums to one. We also assume that the number of potential entrants in bin s is MG_s, so that the total number of potential entrants is M.

For a given number of firms, let us look at the expression for A:

$$\begin{aligned}
A =& \sum_{s=1}^{S} (\varphi^s)^{\frac{1}{1-\alpha}} \mu_s \\
=& (\varphi^{s^*-1})^{\frac{1}{1-\alpha}} \mu_{s^*-1} + \sum_{s=s^*}^{S} (\varphi^s)^{\frac{1}{1-\alpha}} \mu_s \\
=& (\varphi^{s^*-1})^{\frac{1}{1-\alpha}} a \left(MK_eC_1 + MK_eC_2 \left(\varphi^{s^*}\right)^{-\delta_e} + MK_eC_3 + MK_e(\varphi^{s^*})^{-\delta_e} \right) \\
&+ \sum_{s=s^*}^{S} (\varphi^s)^{\frac{1}{1-\alpha}} \left(MK_eC_1 \left(\frac{\varphi^s}{\varphi^{s^*}}\right)^{-\delta} + MK_eC_2 (\varphi^s)^{-\delta_e} + MK_eC_3 \right)
\end{aligned}$$

Dividing both sides by M, we get

$$\begin{aligned}
\frac{A}{M} =& a(\varphi^{s^*-1})^{\frac{1}{1-\alpha}} \left(K_eC_1 + K_eC_2 \left(\varphi^{s^*}\right)^{-\delta_e} + K_eC_3 + K_e(\varphi^{s^*})^{-\delta_e} \right) \\
&+ K_eC_1 \left(\varphi^{s^*}\right)^{\delta} \sum_{s=s^*}^{S} \left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^s + K_eC_2 \sum_{s=s^*}^{S} \left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^s + K_eC_3 \sum_{s=s^*}^{S} (\varphi^{\frac{1}{1-\alpha}})^s \\
=& a(\varphi^{s^*-1})^{\frac{1}{1-\alpha}} \left(K_eC_1 + K_eC_2 \left(\varphi^{s^*}\right)^{-\delta_e} + K_eC_3 + K_e(\varphi^{s^*})^{-\delta_e} \right) \\
&+ K_eC_1 \left(\varphi^{s^*}\right)^{\delta} \frac{\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^{s^*} - \left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^{S+1}}{1-\varphi^{-\delta+\frac{1}{1-\alpha}}} + K_eC_2 \frac{\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{s^*} - \left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{S+1}}{1-\varphi^{-\delta_e+\frac{1}{1-\alpha}}} \\
&+ K_eC_3 \frac{(\varphi^{\frac{1}{1-\alpha}})^{s^*} - (\varphi^{\frac{1}{1-\alpha}})^{S+1}}{1-\varphi^{\frac{1}{1-\alpha}}}
\end{aligned}$$

Recall that under assumption 2, we have

$$\begin{aligned}
(\varphi^{\frac{1}{1-\alpha}})^S &= (\varphi^S)^{\frac{1}{1-\alpha}} = (ZN^{1/\delta})^{\frac{1}{1-\alpha}} = Z^{\frac{1}{1-\alpha}} N^{\frac{1}{\delta(1-\alpha)}} \\
\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^S &= \left(\varphi^S\right)^{-\delta+\frac{1}{1-\alpha}} = \left(ZN^{1/\delta}\right)^{-\delta+\frac{1}{1-\alpha}} = Z^{-\delta+\frac{1}{1-\alpha}} N^{-1+\frac{1}{\delta(1-\alpha)}} \\
\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^S &= \left(\varphi^S\right)^{-\delta_e+\frac{1}{1-\alpha}} = \left(ZN^{1/\delta}\right)^{-\delta_e+\frac{1}{1-\alpha}} = Z^{-\delta_e+\frac{1}{1-\alpha}} N^{-\frac{\delta_e}{\delta}+\frac{1}{\delta(1-\alpha)}}
\end{aligned}$$

Since we assume that $\delta(1-\alpha) > 1$ and $\delta_e(1-\alpha) > 1$ we have both $-\frac{\delta_e}{\delta} + \frac{1}{\delta(1-\alpha)} < 0$ and $-1+\frac{1}{\delta(1-\alpha)} < 0$ and thus both $\left(\varphi^{-\delta+\frac{1}{1-\alpha}}\right)^S$ and $\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^S$ converge to zero when N goes to infinity. We also have that

$$C_3(\varphi^{\frac{1}{1-\alpha}})^S = \frac{-\varphi^{-\delta_e} Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} Z^{\frac{1}{1-\alpha}} N^{-\delta_e/\delta+\frac{1}{\delta(1-\alpha)}} \underset{N\to\infty}{\longrightarrow} 0$$

Putting these results together yields

$$\begin{aligned}
\frac{A}{M} \underset{N\to\infty}{\longrightarrow} A^\infty :=& a(\varphi^{\overline{s^*}-1})^{\frac{1}{1-\alpha}} \left((\varphi^{\delta_e}-1)C_1^\infty + (\varphi^{\delta_e}-1)(C_2+1) \left(\varphi^{\overline{s^*}}\right)^{-\delta_e} \right) \\
&+ (\varphi^{\delta_e}-1)C_1^\infty \frac{\left(\varphi^{\frac{1}{1-\alpha}}\right)^{\overline{s^*}}}{1-\varphi^{-\delta+\frac{1}{1-\alpha}}} + (\varphi^{\delta_e}-1)C_2 \frac{\left(\varphi^{-\delta_e+\frac{1}{1-\alpha}}\right)^{\overline{s^*}}}{1-\varphi^{-\delta_e+\frac{1}{1-\alpha}}}
\end{aligned}$$

In other words, under assumption 2 and if $\delta(1-\alpha) > 1$ and $\delta_e(1-\alpha) > 1$ then $A \underset{M\to\infty}{\sim} A^\infty M$ or

$$A \underset{N\to\infty}{\sim} E^\infty A^\infty N \tag{36}$$

Step 2: How D evolves when the number of incumbents converges to infinity

For a given number of firms, the steady-state value of D:

$$
\begin{aligned}
D =& \sum_{s=1}^{S} \left((\varphi^s)^{\frac{1}{1-\alpha}} \right)^2 \mu_s \\
=& \left((\varphi^{s^*-1})^{\frac{1}{1-\alpha}} \right)^2 \mu_{s^*-1} + \sum_{s=s^*}^{S} \left((\varphi^s)^{\frac{1}{1-\alpha}} \right)^2 \mu_s \\
\frac{D}{M} =& (\varphi^{s^*-1})^{\frac{2}{1-\alpha}} \hat{\mu}_{s^*-1} + K_e C_1 (\varphi^{s^*})^{\delta} \sum_{s=s^*}^{S} (\varphi^s)^{\frac{2}{1-\alpha}-\delta} + K_e C_2 \sum_{s=s^*}^{S} (\varphi^s)^{\frac{2}{1-\alpha}-\delta_e} + K_e C_3 \sum_{s=s^*}^{S} (\varphi^s)^{\frac{2}{1-\alpha}} \\
=& a(\varphi^{s^*-1})^{\frac{2}{1-\alpha}} \left(K_e C_1 + K_e (C_2+1) \left(\varphi^{s^*}\right)^{-\delta_e} + K_e C_3 \right) \\
&+ K_e C_1 (\varphi^{s^*})^{\delta} \frac{(\varphi^{\frac{2}{1-\alpha}-\delta})^{s^*} - (\varphi^{\frac{2}{1-\alpha}-\delta})^{S+1}}{1-\varphi^{\frac{2}{1-\alpha}-\delta}} \\
&+ K_e C_2 \frac{(\varphi^{\frac{2}{1-\alpha}-\delta_e})^{s^*} - (\varphi^{\frac{2}{1-\alpha}-\delta_e})^{S+1}}{1-\varphi^{\frac{2}{1-\alpha}-\delta_e}} \\
&+ K_e C_3 \frac{(\varphi^{\frac{2}{1-\alpha}})^{s^*} - (\varphi^{\frac{2}{1-\alpha}})^{S+1}}{1-\varphi^{\frac{2}{1-\alpha}}}
\end{aligned}
$$

Under assumption 2, we have

$$
\begin{aligned}
(\varphi^{\frac{2}{1-\alpha}-\delta})^S &= (\varphi^S)^{\frac{2}{1-\alpha}-\delta} = (ZN^{1/\delta})^{\frac{2}{1-\alpha}-\delta} = Z^{\frac{2}{1-\alpha}-\delta} N^{\frac{2}{\delta(1-\alpha)}-1} \\
(\varphi^{\frac{2}{1-\alpha}-\delta_e})^S &= (\varphi^S)^{\frac{2}{1-\alpha}-\delta_e} = (ZN^{1/\delta})^{\frac{2}{1-\alpha}-\delta_e} = Z^{\frac{2}{1-\alpha}-\delta_e} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}} \\
C_3(\varphi^{\frac{2}{1-\alpha}})^S &= C_3(\varphi^S)^{\frac{2}{1-\alpha}} = \frac{-\varphi^{-\delta_e} Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} (Z)^{\frac{2}{1-\alpha}} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}}
\end{aligned}
$$

Under the assumption that $\delta(1-\alpha) < 2$ and $\delta_e(1-\alpha) < 2$, these terms diverge when N goes to infinity. Thus we are able to look at the asymptotic equivalent of D/M,

$$
\begin{aligned}
\frac{D}{M} \underset{N\to\infty}{\sim} & a(\varphi^{\overline{s^*}-1})^{\frac{2}{1-\alpha}} \left((\varphi_e^{\delta}-1) C_1^{\infty} + (\varphi_e^{\delta}-1)(C_2+1) \left(\varphi^{\overline{s^*}}\right)^{-\delta_e} \right) \\
&+ (\varphi_e^{\delta}-1) C_1^{\infty} (\varphi^{\overline{s^*}})^{\delta} \frac{-\varphi^{\frac{2}{1-\alpha}-\delta}}{1-\varphi^{\frac{2}{1-\alpha}-\delta}} Z^{\frac{2}{1-\alpha}-\delta} N^{\frac{2}{\delta(1-\alpha)}-1} \\
&+ \left((\varphi_e^{\delta}-1) C_2 \frac{-\varphi^{\frac{2}{1-\alpha}-\delta_e}}{1-\varphi^{\frac{2}{1-\alpha}-\delta_e}} Z^{\frac{2}{1-\alpha}-\delta_e} + (\varphi_e^{\delta}-1) \frac{-\varphi^{\frac{2}{1-\alpha}}}{1-\varphi^{\frac{2}{1-\alpha}}} \frac{-\varphi^{-\delta_e} Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} (Z)^{\frac{2}{1-\alpha}} \right) N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}}
\end{aligned}
$$

By using the intermediate result above on the link between N and M, we have

$$
\begin{aligned}
D \underset{N\to\infty}{\sim} & a(\varphi^{\overline{s^*}-1})^{\frac{2}{1-\alpha}} \left((\varphi_e^{\delta}-1) C_1^{\infty} + (\varphi_e^{\delta}-1)(C_2+1) \left(\varphi^{\overline{s^*}}\right)^{-\delta_e} \right) E^{\infty} N \\
&+ (\varphi_e^{\delta}-1) C_1^{\infty} (\varphi^{\overline{s^*}})^{\delta} \frac{-\varphi^{\frac{2}{1-\alpha}-\delta}}{1-\varphi^{\frac{2}{1-\alpha}-\delta}} Z^{\frac{2}{1-\alpha}-\delta} E^{\infty} N^{\frac{2}{\delta(1-\alpha)}} \\
&+ \left((\varphi_e^{\delta}-1) C_2 \frac{-\varphi^{\frac{2}{1-\alpha}-\delta_e}}{1-\varphi^{\frac{2}{1-\alpha}-\delta_e}} Z^{\frac{2}{1-\alpha}-\delta_e} + (\varphi_e^{\delta}-1) \frac{-\varphi^{\frac{2}{1-\alpha}}}{1-\varphi^{\frac{2}{1-\alpha}}} \frac{-\varphi^{-\delta_e} Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)} (Z)^{\frac{2}{1-\alpha}} \right) E^{\infty} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}+1}
\end{aligned}
$$

Or equivalently, defining the appropriate constants $D_1^{\infty}, D_2^{\infty}$ and D_3^{∞} we have that, under Assumption 2:

$$
D \underset{N\to\infty}{\sim} D_1^{\infty} N + D_2^{\infty} N^{\frac{2}{\delta(1-\alpha)}} + D_3^{\infty} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}+1} \tag{37}
$$

Step 3: How D/A^2 evolves with N:

The first term of aggregate volatility described by Equation 33 is $\frac{D}{A^2}$. Let us look at its equivalent when N goes to infinity by combining Equations 36 and 37

$$\frac{D}{A^2} \underset{N\to\infty}{\sim} \frac{\frac{D_1^\infty}{(E^\infty A^\infty)}}{N} + \frac{\frac{D_2^\infty}{(E^\infty A^\infty)}}{N^{2-\frac{2}{\delta(1-\alpha)}}} + \frac{\frac{D_3^\infty}{(E^\infty A^\infty)}}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}}$$

Under the assumptions that $\delta(1-\alpha) < 2$ and $\delta_e(1-\alpha) < 2$, then $2 - \frac{2}{\delta(1-\alpha)} < 1$ and $1 + \frac{\delta_e}{\delta} - \frac{2}{\delta(1-\alpha)} < 1$. In other words, the last two terms dominate the first term and thus:

$$\frac{D}{A^2} \underset{N\to\infty}{\sim} \frac{\frac{D_2^\infty}{(E^\infty A^\infty)}}{N^{2-\frac{2}{\delta(1-\alpha)}}} + \frac{\frac{D_3^\infty}{(E^\infty A^\infty)}}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}} \tag{38}$$

□

Step 4: How $E(\varphi^2)$ and O^σ evolve with N

Here we prove a similar result for the remaining terms in Equation 33, i.e. $E(\varphi^2)/A^2$ and O^σ/A^2. We first find the expression for $\frac{E(\varphi^2)}{M}$ and then for $\frac{O^\sigma}{M}$, when $N \to \infty$. The steady-state expression of $E(\varphi^2)$ is

$$\begin{aligned}
E(\varphi^2) &= \left(M \sum_{s=s^*}^{S} G_s \left(\varphi^{2s}\right)^{\frac{1}{1-\alpha}} \right) - \left(\left(\varphi^{2(s^*-1)}\right)^{\frac{1}{1-\alpha}} \mu_{s^*-1,t} \right) \\
&= \left(M K_e \sum_{s=s^*}^{S} (\varphi^s)^{-\delta_e} \left(\varphi^{2s}\right)^{\frac{1}{1-\alpha}} \right) - \left(\left(\varphi^{2(s^*-1)}\right)^{\frac{1}{1-\alpha}} \mu_{s^*-1,t} \right) \\
&= M K_e \frac{\left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right)^{S+1} - \left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right)^{s^*}}{\left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - 1} - \left(\left(\varphi^{\frac{2}{1-\alpha}}\right)^{(s^*-1)} \mu_{s^*-1,t} \right)
\end{aligned}$$

Under Assumption 2, we still have

$$(\varphi^{\frac{2}{1-\alpha}-\delta_e})^S = (\varphi^S)^{\frac{2}{1-\alpha}-\delta_e} = (ZN^{1/\delta})^{\frac{2}{1-\alpha}-\delta_e} = Z^{\frac{2}{1-\alpha}-\delta_e} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}}$$

Thus, it follows

$$\begin{aligned}
\frac{E(\varphi^2)}{M} &= K_e \frac{\left(Z^{\frac{2}{1-\alpha}-\delta_e} N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}} \varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - \left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right)^{s^*}}{\left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - 1} \\
&\quad - \left(\left(\varphi^{\frac{2}{1-\alpha}}\right)^{(s^*-1)} \left(K_e C_1 + K_e (C_2+1)\varphi^{s^*} + K_e C_3\right) \right) \\
&= K_e N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}} \frac{\left(Z^{\frac{2}{1-\alpha}-\delta_e} \varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - N^{\frac{-2}{\delta(1-\alpha)}+\frac{\delta_e}{\delta}} \left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right)^{s^*}}{\left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - 1} \\
&\quad - \left(\left(\varphi^{\frac{2}{1-\alpha}}\right)^{(s^*-1)} \left(K_e C_1 + K_e (C_2+1)\varphi^{s^*} + K_e C_3\right) \right)
\end{aligned}$$

Under the assumption that $\delta_e(1-\alpha) < 2$, we have

$$\frac{E(\varphi^2)}{M} \sim K_e N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}} \frac{\left(Z^{\frac{2}{1-\alpha}-\delta_e} \varphi^{-\delta_e+\frac{2}{1-\alpha}}\right)}{\left(\varphi^{-\delta_e+\frac{2}{1-\alpha}}\right) - 1} - \left(\left(\varphi^{\frac{2}{1-\alpha}}\right)^{(\overline{s^*}-1)} \left(\varphi^{\delta_e} - 1\right) \left(C_1^\infty + (C_2+1)\varphi^{\overline{s^*}}\right) \right)$$

Recall that $M \sim E^\infty N$. Then, for some constant E_1^∞ and E_2^∞, we have $E(\varphi^2) \sim E_1^\infty N^{1-\frac{\delta_e}{\delta}+\frac{2}{\delta(1-\alpha)}} + E_2^\infty N$. Using the fact that $A^2 \underset{N\to\infty}{\sim} E^\infty A^\infty N$ and the above equation, we get for some other constant $\mathcal{E}_1^\infty$ and $\mathcal{E}_2^\infty$:

$$\frac{E(\varphi^2)}{A^2} \sim \frac{\mathcal{E}_1^\infty}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}} + \frac{\mathcal{E}_2^\infty}{N} \sim \frac{\mathcal{E}_1^\infty}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}} \tag{39}$$

where the last equivalence comes from the fact that $\delta(1-\alpha) > 2$ and $\delta_e(1-\alpha) > 2$ and thus $1 > 1 + \frac{\delta_e}{\delta} - \frac{2}{\delta(1-\alpha)}$.

The steady-state expression for O^σ is:

$$\begin{aligned}\frac{O^\sigma}{M} &= -K_e(\varrho-\varrho')(\varphi^{-\delta_e}(\varphi^{\frac{1}{1-\alpha}})^2)^S - (\varrho-\varrho')(\varphi^{\frac{1}{1-\alpha}})^{2S}\hat{\mu}_S \\ &= -K_e(\varrho-\varrho')(\varphi^{-\delta_e}(\varphi^{\frac{1}{1-\alpha}})^2)^S - (\varrho-\varrho')(\varphi^{\frac{1}{1-\alpha}})^{2S}\left(K_eC_1(\varphi^{s^*})^{-\delta}(\varphi^S)^{-\delta} + K_eC_2(\varphi^S)^{-\delta_e} + K_eC_3\right) \\ &= -K_e(\varrho-\varrho')(\varphi^{-\delta_e+\frac{2}{1-\alpha}})^S - (\varrho-\varrho')\left(K_eC_1(\varphi^{s^*})^{-\delta}(\varphi^{-\delta+\frac{2}{1-\alpha}})^S + K_eC_2(\varphi^{-\delta_e+\frac{2}{1-\alpha}})^S + K_eC_3(\varphi^{\frac{2}{1-\alpha}})^S\right)\end{aligned}$$

Recall that under Assumption 2,

$$\begin{aligned}(\varphi^{\frac{2}{1-\alpha}-\delta})^S &= (\varphi^S)^{\frac{2}{1-\alpha}-\delta} = (ZN^{1/\delta})^{\frac{2}{1-\alpha}-\delta} = Z^{\frac{2}{1-\alpha}-\delta}N^{\frac{2}{\delta(1-\alpha)}-1} \\ (\varphi^{\frac{2}{1-\alpha}-\delta_e})^S &= (\varphi^S)^{\frac{2}{1-\alpha}-\delta_e} = (ZN^{1/\delta})^{\frac{2}{1-\alpha}-\delta_e} = Z^{\frac{2}{1-\alpha}-\delta_e}N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}} \\ C_3(\varphi^{\frac{2}{1-\alpha}})^S &= C_3(\varphi^S)^{\frac{2}{1-\alpha}} = \frac{-\varphi^{-\delta_e}Z^{-\delta_e}}{(1-\varphi^{-\delta_e})(a-c)}(Z)^{\frac{2}{1-\alpha}}N^{\frac{2}{\delta(1-\alpha)}-\frac{\delta_e}{\delta}}\end{aligned}$$

Using the above relations, we then have, for some constants O_1^∞ and O_2^∞,

$$O^\sigma \sim O_1^\infty N^{1-\frac{\delta_e}{\delta}+\frac{2}{\delta(1-\alpha)}} + O_2^\infty N^{\frac{2}{\delta(1-\alpha)}}$$

from which it follows that for, some other constants, $\mathcal{O}_1^\infty$ and $\mathcal{O}_2^\infty$

$$\frac{O^\sigma}{A^2} \sim \frac{\mathcal{O}_1^\infty}{N^{1+\frac{\delta_e}{\delta}-\frac{2}{\delta(1-\alpha)}}} + \frac{\mathcal{O}_2^\infty}{N^{2-\frac{2}{\delta(1-\alpha)}}} \tag{40}$$

Putting Equations 38, 39 and 40 together yields the results in Equation 13. □

B.9 Proof of Proposition 6

To solve for the general case with aggregate uncertainty, we deploy a different strategy relative to that used in the stationary case. Whereas we used a constructive proof for the stationary case, we follow a guess and verify strategy for the case featuring aggregate fluctuations. We first show some useful preliminary results to compute conditional expectations. We then show that the value function has to be bounded above by the value of a firm when $c_f = 0$. Finally, we form our guess and solve for the value function.

B.9.1 Preliminary Results

Lemma 3 *Under Assumption 3, for any* ξ

$$\mathbb{E}_t\left[w_{t+1}^\xi\right] \approx w_t^\xi \rho^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} I(\xi)$$

where

$$I(\xi) = \int_{-\infty}^{\infty}\left(1 + \frac{E(\varphi)}{A} + \frac{O^A}{A} + \frac{\sigma}{A}\varepsilon\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \phi(\varepsilon)d\varepsilon$$

where $\phi(\varepsilon)$ *is the probability distribution function of a standard normal random variable and* X *is the stationary equilibrium value of* X_t.

Proof: First note that, in equilibrium, $w_t^\xi = \left(\alpha^{\frac{1}{1-\alpha}}\frac{A_t}{M}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}}$. Let us now compute the conditional expectation

$$\begin{aligned}
\mathbb{E}_t\left[w_{t+1}^\xi\right] &= \int_{\mu_{t+1}} w_{t+1}^\xi \Gamma(d\mu_{t+1}|\mu_t) = \int_{\mu_{t+1}} \left(\alpha^{\frac{1}{1-\alpha}}\frac{A_{t+1}}{M}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \Gamma(d\mu_{t+1}|\mu_t) \\
&= \left(\alpha^{\frac{1}{1-\alpha}}\frac{A_t}{M}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \int_{\mu_{t+1}} \left(\frac{A_{t+1}}{A_t}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \Gamma(d\mu_{t+1}|\mu_t) \\
&= \left(\alpha^{\frac{1}{1-\alpha}}\frac{A_t}{M}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \int_{\mu_{t+1}} \left(\frac{\rho A_t + \rho E_t(\varphi) + O_t^A + \sigma_t\varepsilon_{t+1}}{A_t}\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \Gamma(d\mu_{t+1}|\mu_t) \\
&= \rho^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} w_t^\xi \int_{-\infty}^{\infty} \left(1 + \frac{E_t(\varphi)}{A_t} + \frac{O_t^A}{\rho A_t} + \frac{\sigma_t}{\rho A_t}\varepsilon\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \phi(\varepsilon)d\varepsilon
\end{aligned}$$

where we use Theorem 3 in the third line. Under Assumption 3, the integral in the last equation is equal to $I(\xi)$ which completes the proof of the lemma. □

B.9.2 Bounded Above by the case $c_f = 0$

Lemma 4 *For $S \to \infty$, the value function of a firm at productivity level φ^s with aggregate state μ_t satisfies the following inequality*

$$V(\mu_t, \varphi^s) \leq V^{c_f=0}(\mu_t, \varphi^s)$$

where $V^{c_f=0}(\mu_t, \varphi^s)$ is the value of a firm at productivity level φ^s with aggregate state μ_t that faces an operating cost c_f equal to zero. This is equal to

$$V^{c_f=0}(\mu_t, \varphi^s) = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}} (\varphi^s)^{\frac{1}{1-\alpha}}$$

where $\widetilde{\beta}_\alpha = \beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}$ and $I(\xi) = \int_{-\infty}^{\infty}\left(1 + \frac{E(\varphi)}{A} + \frac{O^A}{A} + \frac{\sigma}{A}\varepsilon\right)^{\frac{(1-\alpha)\xi}{\gamma(1-\alpha)+1}} \phi(\varepsilon)d\varepsilon$. The inequality becomes an equality when $c_f = 0$.

Proof:
We prove this proposition in two steps. We first show the inequality stated in the Lemma and then solve for $V^{c_f=0}(\mu_t, \varphi^s)$.

Bounding $V(\mu_t, \varphi^s) \leq V^{c_f=0}(\mu_t, \varphi^s)$: First note that the instantaneous profit is bounded above by the profit of a firm facing zero fixed operating costs c_f:

$$\pi^*(\mu, \varphi^s) = (\varphi^s)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) - c_f \leq \pi^{c_f=0}(\mu, \varphi^s) = (\varphi^s)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha)$$

A firm j's problem can be rewritten as a stopping time problem:

$$V(\mu_t, \varphi^{s_{j,t}}) = \max_L \left\{\mathbb{E}_t \sum_{i=t}^{L} \beta^{i-t}\pi^*(\mu_{i+t}, \varphi^{s_{j,t+i}})\right\}$$

where the j firm choose the optimal time of exit, L, to maximize its discounted sum of instantaneous profit. The same firm facing an operating cost $c_f = 0$ every period will have a value

$$V^{c_f=0}(\mu_t, \varphi^{s_{j,t}}) = \max_L \left\{ \mathbb{E}_t \sum_{i=t}^{L} \beta^{i-t} \pi^{c_f=0}(\mu_{i+t}, \varphi^{s_{j,t+i}}) \right\}$$

It is optimal for this firm to choose $L = \infty$. Since $\forall (s, \mu), \pi^*(\mu, \varphi^s) \leq \pi^{c_f=0}(\mu, \varphi^s)$ we have

$$V(\mu_t, \varphi^{s_{j,t}}) \leq V^{c_f=0}(\mu_t, \varphi^{s_{j,t}})$$

This completes the first part of the proof.

Solving for $V^{c_f=0}(\mu_t, \varphi^s)$: Note that $V^{c_f=0}(\mu_t, \varphi^s)$ must satisfy the following Bellman equation:

$$V^{c_f=0}(\mu_t, \varphi^{s_{j,t}}) = \pi^{c_f=0}(\mu, \varphi^{s_{j,t}}) + \beta \mathbb{E}_t \left[V^{c_f=0}(\mu_t, \varphi^{s_{j,t+1}}) \right] \tag{41}$$

We are following a guess and verify strategy. Our guess is

$$V^{c_f=0}(\mu_t, \varphi^s) = K_1 + K_2 w_t^{\frac{-\alpha}{1-\alpha}} (\varphi^s)^{\frac{1}{1-\alpha}}$$

and we are solving for K_1 and K_2. Let us compute the right hand side of the Bellman equation above. It is easy to show using the definition of ρ

$$aV^{c_f=0}(\mu_t, \varphi^{s-1}) + bV^{c_f=0}(\mu_t, \varphi^s) + cV^{c_f=0}(\mu_t, \varphi^{s+1}) = K_1 + K_2 \rho w_t^{\frac{-\alpha}{1-\alpha}} (\varphi^s)^{\frac{1}{1-\alpha}}$$

and the continuation value is

$$\begin{aligned} \int_{w'} \left(aV^{c_f=0}(\mu', \varphi^{s-1}) + bV^{c_f=0}(\mu', \varphi^s) + cV^{c_f=0}(\mu', \varphi^{s+1}) \right) \Gamma(d\mu' | \mu_t) \\ = K_1 + K_2 \rho (\varphi^s)^{\frac{1}{1-\alpha}} \int_{w'} w'^{\frac{-\alpha}{1-\alpha}} \Gamma(d\mu' | \mu_t) \\ = K_1 + K_2 \rho (\varphi^s)^{\frac{1}{1-\alpha}} w_t^{\frac{-\alpha}{1-\alpha}} I \left(\frac{-\alpha}{1-\alpha} \right) \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}} \end{aligned}$$

where we use Lemma 3 in the last line of derivations. The Bellman Equation 41 writes

$$K_1 + K_2 w_t^{\frac{-\alpha}{1-\alpha}} (\varphi^s)^{\frac{1}{1-\alpha}} = (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w_t} \right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) + \beta K_1 + \beta K_2 \rho (\varphi^s)^{\frac{1}{1-\alpha}} w_t^{\frac{-\alpha}{1-\alpha}} I \left(\frac{-\alpha}{1-\alpha} \right) \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}$$

Matching coefficients yields

$$\begin{aligned} K_1 &= \beta K_1 \\ K_2 &= \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1 - \rho \beta I \left(\frac{-\alpha}{1-\alpha} \right) \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} \end{aligned}$$

Since $\beta < 1$ it follows that $K_1 = 0$ and the value of a firm facing zero operating cost at productivity level φ^s and aggregate state μ_t is equal to

$$V^{c_f=0}(\mu_t, \varphi^s) = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1 - \rho \beta I \left(\frac{-\alpha}{1-\alpha} \right) \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}} (\varphi^s)^{\frac{1}{1-\alpha}}$$

□

Proof of the proposition

The value of an incumbent firm, $V(\mu_t,\varphi^s)$, satisfies the following Bellman equation:

$$V(\mu_t,\varphi^s)=\pi^*(\mu_t,\varphi^s)+\beta\max\left\{0,\int_{\mu'}\left(aV(\mu',\varphi^{s-1})+bV(\mu',\varphi^s)+cV(\mu',\varphi^{s+1})\right)\Gamma(d\mu'|\mu_t)\right\}$$

the policy function of such a problem satisfies a threshold rule, with threshold $s^*(\mu)$ such that

$$V(\mu_t,\varphi^s)=\left\{\begin{array}{cr}\pi^*(\mu_t,\varphi^s)+\beta\int_{\mu'}\left(aV(\mu',\varphi^{s-1})+bV(\mu',\varphi^s)+cV(\mu',\varphi^{s+1})\right)\Gamma(d\mu'|\mu_t) & \text{for } s\geq s^*(\mu_t)\\ \pi^*(\mu_t,\varphi^s) & \text{for } s\leq s^*(\mu_t)-1\end{array}\right. \quad (42)$$

We adopt a guess and verify strategy to prove this proposition. In this case, we are forming a guess for both $s^*(\mu_t)$ and $V(\mu_t,\varphi^s)$. To form our guess we are going to draw our inspiration from the stationary case. In that case, we first solved for the homogeneous equation, and we were using the roots of this equation. The equivalent of this homogeneous equation in the current setting is:

$$a+bX+cX^2=\frac{X}{\beta\rho^{\frac{-\alpha(1-\alpha)}{\gamma(1-\alpha)+1}\frac{\log X}{\log\varphi}}I\left(-\alpha\frac{\log X}{\log\varphi}\right)}$$

Let $\widetilde{r_1}$ and $\widetilde{r_2}$ be the two solutions of this equation, such that $\widetilde{r_1}>\varphi^{\frac{1}{1-\alpha}}>\widetilde{r_2}$. Let us define the constants $\widetilde{\beta_i}=\beta\rho^{\frac{-\alpha(1-\alpha)}{\gamma(1-\alpha)+1}\frac{\log\widetilde{r_i}}{\log\varphi}}I\left(-\alpha\frac{\log\widetilde{r_i}}{\log\varphi}\right)$ for $i=1,2$. It is clear that $\widetilde{r_i}$ satisfies

$$a\widetilde{r_i}^s+b\widetilde{r_i}^{s+1}+c\widetilde{r_i}^{s+2}=\widetilde{r_i}^s\left(a+b\widetilde{r_i}+c\widetilde{r_i}^2\right)=\widetilde{r_i}^s\frac{\widetilde{r_i}}{\widetilde{\beta_i}}=\frac{\widetilde{r_i}^{s+1}}{\widetilde{\beta_i}}$$

Guess for $s^*(\mu_t)$: We are guessing that the entry/exit thesholds take the same form as in the stationary case:

$$s^*(\mu_t)=(1-\alpha)\frac{\log\chi}{\log\varphi}+\alpha\frac{\log w_t}{\log\varphi}$$

where χ is a constant to be solved for. Given this, it is easy to show that for any $X>0$

$$X^{-s^*(w_t)}=X^{-(1-\alpha)\frac{\log\chi}{\log\varphi}-\alpha\frac{\log w_t}{\log\varphi}}=X^{-(1-\alpha)\frac{\log\chi}{\log\varphi}}X^{-\alpha\frac{\log w_t}{\log\varphi}}=\chi^{-(1-\alpha)\frac{\log X}{\log\varphi}}w_t^{-\alpha\frac{\log X}{\log\varphi}}$$

Guess for $V(\mu_t,\varphi^s)$: To form a guess of the value function, we draw inspiration from the stationary case and thus our guess is, for $s\geq s^*(w_t)$

$$V(\mu_t,\varphi^s)=K_1+K_2w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+K_3\widetilde{r_2}^{s+1-s^*(w_t)}+K_4\widetilde{r_1}^{s+1-s^*(w_t)}$$

where the constants K_1,K_2,K_3 and K_4 have to be solves for. Using this guess for $s^*(w_t)$ gives

$$V(\mu_t,\varphi^s)=K_1+K_2w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+K_3\chi^{-(1-\alpha)\frac{\log\widetilde{r_2}}{\log\varphi}}w_t^{-\alpha\frac{\log\widetilde{r_2}}{\log\varphi}}\widetilde{r_2}^{s+1}+K_4\chi^{-(1-\alpha)\frac{\log\widetilde{r_1}}{\log\varphi}}w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}^{s+1}$$

Let us introduce the following simplifying notation. Let us define $\widetilde{K_3}=K_3\chi^{-(1-\alpha)\frac{\log\widetilde{r_2}}{\log\varphi}}$ and $\widetilde{K_4}=K_4\chi^{-(1-\alpha)\frac{\log\widetilde{r_1}}{\log\varphi}}$, and $V(w_t,s)=V(\mu_t,\varphi^s)$. With this notation, our guess can be written, for $s\geq s^*(w_t)$

$$V(w_t,s)=K_1+K_2w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+\widetilde{K_3}w_t^{-\alpha\frac{\log\widetilde{r_2}}{\log\varphi}}\widetilde{r_2}^{s+1}+\widetilde{K_4}w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}^{s+1}$$

Bellman equation: We are computing the right hand side of the Bellman Equation 42 starting with the continuation value of an incumbent firm. Note that

$$
\begin{aligned}
aV(w_t, s-1) + bV(w_t, s) + cV(w_t, s+1) = \\
& K_1(a+b+c) \\
& + K_2 w_t^{\frac{-\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^s \left(a\varphi^{\frac{-1}{1-\alpha}} + b + c\varphi^{\frac{1}{1-\alpha}}\right) \\
& + \widetilde{K_3} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \left(a\widetilde{r_2}^s + b\widetilde{r_2}^{s+1} + c\widetilde{r_2}^{s+2}\right) \\
& + \widetilde{K_4} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \left(a\widetilde{r_1}^s + b\widetilde{r_1}^{s+1} + c\widetilde{r_1}^{s+2}\right) \\
=& K_1 + K_2 \rho w_t^{\frac{-\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^s + \widetilde{K_3} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \frac{1}{\widetilde{\beta_2}} \widetilde{r_2}^{s+1} + \widetilde{K_4} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \frac{1}{\widetilde{\beta_1}} \widetilde{r_1}^{s+1}
\end{aligned}
$$

using the definition of ρ and $\widetilde{r_i}$. Let us now compute the continuation value of an incumbent

$$
\begin{aligned}
& \int_{w'} \left[aV(w', s-1) + bV(w', s) + cV(w', s+1)\right] \Gamma(d\mu'|\mu_t) \\
&= K_1 + K_2\rho \left(\varphi^{\frac{1}{1-\alpha}}\right)^s \int_{w'} w'^{\frac{-\alpha}{1-\alpha}} \Gamma(d\mu'|\mu_t) + \widetilde{K_3} \frac{1}{\widetilde{\beta_2}} \widetilde{r_2}^{s+1} \int_{w'} w'^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \Gamma(d\mu'|\mu_t) + \widetilde{K_4} \frac{1}{\widetilde{\beta_1}} \widetilde{r_1}^{s+1} \int_{w'} w'^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \Gamma(d\mu'|\mu_t) \\
&= K_1 + K_2\rho \left(\varphi^{\frac{1}{1-\alpha}}\right)^s I\left(\frac{-\alpha}{1-\alpha}\right) w_t^{\frac{-\alpha}{1-\alpha}} \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}} \\
&\quad + \widetilde{K_3} \frac{1}{\widetilde{\beta_2}} \widetilde{r_2}^{s+1} I\left(-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}\right) w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \rho^{\frac{-\alpha(1-\alpha)}{\gamma(1-\alpha)+1} \frac{\log \widetilde{r_2}}{\log \varphi}} + \widetilde{K_4} \frac{1}{\widetilde{\beta_1}} \widetilde{r_1}^{s+1} I\left(-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}\right) w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \rho^{\frac{-\alpha(1-\alpha)}{\gamma(1-\alpha)+1} \frac{\log \widetilde{r_1}}{\log \varphi}} \\
&= K_1 + K_2\rho \left(\varphi^{\frac{1}{1-\alpha}}\right)^s I\left(\frac{-\alpha}{1-\alpha}\right) w_t^{\frac{-\alpha}{1-\alpha}} \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}} + \widetilde{K_3} \frac{1}{\beta} \widetilde{r_2}^{s+1} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} + \widetilde{K_4} \frac{1}{\beta} \widetilde{r_1}^{s+1} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}}
\end{aligned}
$$

where we use Lemma 3 and the definition of $\widetilde{\beta_i} = \beta \rho^{\frac{-\alpha(1-\alpha)}{\gamma(1-\alpha)+1} \frac{\log \widetilde{r_i}}{\log \varphi}} I\left(-\alpha \frac{\log \widetilde{r_i}}{\log \varphi}\right)$. We can now write the Bellman equation for $s \geq s^*(w_t)$:

$$
\begin{aligned}
& V(w_t, s) = K_1 + K_2 w_t^{\frac{-\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^s + \widetilde{K_3} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \widetilde{r_2}^{s+1} + \widetilde{K_4} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \widetilde{r_1}^{s+1} = \\
& (\varphi^s)^{\frac{1}{1-\alpha}} \left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}} (1-\alpha) - c_f \\
& \quad + \beta K_1 + K_2 \beta\rho \left(\varphi^{\frac{1}{1-\alpha}}\right)^s I\left(\frac{-\alpha}{1-\alpha}\right) w_t^{\frac{-\alpha}{1-\alpha}} \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}} + \widetilde{K_3} \widetilde{r_2}^{s+1} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} + \widetilde{K_4} \widetilde{r_1}^{s+1} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}}
\end{aligned}
$$

which yields (after simplification and matching coefficients)

$$
\left\{ \begin{array}{ll} K_1 & = -c_f + \beta K_1 \\ K_2 & = K_2 \beta\rho I\left(\frac{-\alpha}{1-\alpha}\right) \rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}} + (1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}} \end{array} \right. \quad \Leftrightarrow \quad \left\{ \begin{array}{ll} K_1 & = \frac{-c_f}{1-\beta} \\ K_2 & = \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\beta\rho I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} \end{array} \right.
$$

We are then left to solve for K_3 and K_4 with the following guess

$$
V(w_t, s) = \frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\beta\rho I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}} \left(\varphi^{\frac{1}{1-\alpha}}\right)^s + \widetilde{K_3} w_t^{-\alpha \frac{\log \widetilde{r_2}}{\log \varphi}} \widetilde{r_2}^{s+1} + \widetilde{K_4} w_t^{-\alpha \frac{\log \widetilde{r_1}}{\log \varphi}} \widetilde{r_1}^{s+1}
$$

Solving for K_4: To solve for K_4, we are using Lemma 4.

$$V(s^*(\mu_t), w_t) \leq \frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\beta\rho I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s + \widetilde{K_3} w_t^{-\alpha\frac{\log\widetilde{r_2}}{\log\varphi}}\widetilde{r_2}^{s+1} + \widetilde{K_4} w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}^{s+1}$$

$$\leq \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^s\right)^{\frac{1}{1-\alpha}}$$

where the first equality comes from the fact that $V(s, w_t)$ is increasing in s for a given w_t and the second inequality from Lemma 4. Let us divide both sides of this inequality by $(\varphi^s)^{\frac{1}{1-\alpha}}$

$$\frac{V(s^*(\mu_t), w_t)}{(\varphi^s)^{\frac{1}{1-\alpha}}} \leq \frac{-c_f}{1-\beta}\frac{1}{(\varphi^s)^{\frac{1}{1-\alpha}}} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\beta\rho I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}} + \widetilde{K_3} w_t^{-\alpha\frac{\log\widetilde{r_2}}{\log\varphi}}\widetilde{r_2}\left(\frac{\widetilde{r_2}}{\varphi^{\frac{1}{1-\alpha}}}\right)^s + \widetilde{K_4} w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}\left(\frac{\widetilde{r_1}}{\varphi^{\frac{1}{1-\alpha}}}\right)^s$$

$$\leq \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}}$$

Since $\widetilde{r_2} < \varphi^{\frac{1}{1-\alpha}} < \widetilde{r_1}$ and $\varphi^{\frac{1}{1-\alpha}} > 1$, for $s \to \infty$ this inequality becomes

$$0 \leq 0 + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\beta\rho I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}} + 0 + \lim_{s\to\infty}\widetilde{K_4} w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}\left(\frac{\widetilde{r_1}}{\varphi^{\frac{1}{1-\alpha}}}\right)^s$$

$$\leq \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}} w_t^{\frac{-\alpha}{1-\alpha}}$$

which implies that $\lim_{s\to\infty}\widetilde{K_4} w_t^{-\alpha\frac{\log\widetilde{r_1}}{\log\varphi}}\widetilde{r_1}\left(\frac{\widetilde{r_1}}{\varphi^{\frac{1}{1-\alpha}}}\right)^s = 0$ and, thus, that $K_4 = 0$ since $\varphi^{\frac{1}{1-\alpha}} < \widetilde{r_1}$. We are thus left to solve for K_3 with the guess

$$V(w_t, s) = \frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s + \widetilde{K_3} w_t^{-\alpha\frac{\log\widetilde{r_2}}{\log\varphi}}\widetilde{r_2}^{s+1}$$

where $\widetilde{\beta}_\alpha = \beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}$.

Solving for K_3: To solve for K_3 we are using the Bellman Equation 42 at $s^*(w_t)$:

$$\begin{aligned}
&aV(w_t, s_t^* - 1) + bV(w_t, s_t^*) + cV(w_t, s_t^* + 1) = \\
&= a\left(\left(\varphi^{s_t^*-1}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) - c_f\right) \\
&\quad + b\left(\frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*} + \widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s_t^*+1}\right) \\
&\quad + c\left(\frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*+1} + \widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s*_t+2}\right) \\
&= \frac{-c_f}{1-\beta}\left(a(1-\beta) + b + c\right) \\
&\quad + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*}\left(a\varphi^{\frac{-1}{1-\alpha}}\left(1-\rho\widetilde{\beta}_\alpha\right) + b + c\varphi^{\frac{1}{1-\alpha}}\right) \\
&\quad + \widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s_t^*}\left(b\widetilde{r_2} + c\widetilde{r_2}^2\right) \\
&= \frac{-c_f}{1-\beta}\left(1 - a\beta\right) \\
&\quad + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*}\left(\rho - a\varphi^{\frac{-1}{1-\alpha}}\rho\widetilde{\beta}_\alpha\right) \\
&\quad + \widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s_t^*}\left(\frac{\widetilde{r_2}}{\widetilde{\beta_2}} - a\right)
\end{aligned}$$

Note that $\widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s*_t} = K_3 \chi^{-(1-\alpha)\frac{\log \widetilde{r_2}}{\log\varphi}} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s_t^*} = K_3 \widetilde{r_2}^{-s_t^*}\widetilde{r_2}^{s_t^*} = K_3$ and that $\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*} = \chi^{(1-\alpha)\frac{\log\varphi^{\frac{1}{1-\alpha}}}{\log\varphi}} w_t^{\alpha\frac{\log\varphi^{\frac{1}{1-\alpha}}}{\log\varphi}} = \chi w_t^{\frac{\alpha}{1-\alpha}}$. With these in hand it follows

$$\begin{aligned}
&aV(w_t, s_t^* - 1) + bV(w_t, s_t^*) + cV(w_t, s_t^* + 1) = \\
&= \frac{-c_f}{1-\beta}(1-a\beta) + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\chi\left(\rho - a\varphi^{\frac{-1}{1-\alpha}}\rho\widetilde{\beta}_\alpha\right) + K_3\left(\frac{\widetilde{r_2}}{\widetilde{\beta_2}} - a\right)
\end{aligned}$$

Note that the above expression is independent of w_t. The Bellman Equation 42 at $s = s_t^*$ is

$$\begin{aligned}
&\frac{-c_f}{1-\beta} + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha} w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s_t^*} + \widetilde{K_3} w_t^{-\alpha\frac{\log \widetilde{r_2}}{\log\varphi}} \widetilde{r_2}^{s_t^*+1} \\
&= \left(\varphi^{s_t^*}\right)^{\frac{1}{1-\alpha}}\left(\frac{\alpha}{w_t}\right)^{\frac{\alpha}{1-\alpha}}(1-\alpha) - c_f \\
&\quad + \frac{-c_f\beta}{1-\beta}(1-a\beta) + \frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\beta\chi\left(\rho - a\varphi^{\frac{-1}{1-\alpha}}\rho\widetilde{\beta}_\alpha\right) + K_3\beta\left(\frac{\widetilde{r_2}}{\widetilde{\beta_2}} - a\right)
\end{aligned}$$

which after simplification yields

$$\begin{aligned}
&\frac{-c_f}{1-\beta}+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\chi+K_3\widetilde{r}_2\\
&=\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)-c_f+\frac{-c_f\beta}{1-\beta}(1-a\beta)+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\beta\chi\left(\rho-a\varphi^{\frac{-1}{1-\alpha}}\rho\widetilde{\beta}_\alpha\right)+K_3\beta\left(\frac{\widetilde{r}_2}{\widetilde{\beta}_2}-a\right)\\
&\Leftrightarrow\\
&K_3\left(\widetilde{r}_2-\widetilde{r}_2\frac{\beta}{\widetilde{\beta}_2}+\beta a\right)=\frac{c_f}{1-\beta}a\beta^2+\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\left(1+\frac{\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha)}{1-\rho\widetilde{\beta}_\alpha}-\frac{1}{1-\rho\widetilde{\beta}_\alpha}\right)\\
&\Leftrightarrow\\
&K_3\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)=\frac{c_f}{1-\beta}a\beta^2+\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{1-\rho\widetilde{\beta}_\alpha}\\
&\Leftrightarrow\\
&K_3=\frac{\frac{c_f}{1-\beta}a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}+\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\left(1-\rho\widetilde{\beta}_\alpha\right)\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}
\end{aligned}$$

where $\widetilde{\beta}_\alpha=\beta I\left(\frac{-\alpha}{1-\alpha}\right)\rho^{\frac{-\alpha}{\gamma(1-\alpha)+1}}$. It follows that the value of an incumbent for $s\geq s_t^*$ is

$$\begin{aligned}
V(w_t,s)=&\frac{-c_f}{1-\beta}+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+\frac{\frac{c_f}{1-\beta}a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}\chi^{-(1-\alpha)\frac{\log\widetilde{r}_2}{\log\varphi}}w_t^{-\alpha\frac{\log\widetilde{r}_2}{\log\varphi}}\widetilde{r}_2^{\,s+1}\\
&+\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\left(1-\rho\widetilde{\beta}_\alpha\right)\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}\chi^{-(1-\alpha)\frac{\log\widetilde{r}_2}{\log\varphi}}w_t^{-\alpha\frac{\log\widetilde{r}_2}{\log\varphi}}\widetilde{r}_2^{\,s+1}
\end{aligned}$$

or equivalently

$$\begin{aligned}
V(w_t,s)=&\frac{-c_f}{1-\beta}+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+\frac{\frac{c_f}{1-\beta}a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}\widetilde{r}_2^{\,s+1-s^*(w_t)}\\
&+\chi\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\left(1-\rho\widetilde{\beta}_\alpha\right)\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}\widetilde{r}_2^{\,s+1-s^*(w_t)}
\end{aligned}$$

which, after rearranging terms, yields

$$\begin{aligned}
&V(w_t,s)=\frac{-c_f}{1-\beta}\left(1-\frac{a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}\widetilde{r}_2^{\,s+1-s^*(w_t)}\right)\\
&+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+\frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{1-\rho\widetilde{\beta}_\alpha}\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s^*(w_t)}\widetilde{r}_2^{\,s+1-s^*(w_t)}\\
&\text{or}\\
&V(w_t,s)=\frac{-c_f}{1-\beta}\left(1-\frac{a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}\widetilde{r}_2^{\,s+1-s^*(w_t)}\right)\\
&+\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s+\frac{\alpha^{\frac{\alpha}{1-\alpha}}(1-\alpha)}{1-\rho\widetilde{\beta}_\alpha}\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s+1}\left(\frac{\widetilde{r}_2}{\varphi^{\frac{1}{1-\alpha}}}\right)^{s+1-s^*(w_t)}\\
&\text{or}\\
&V(w_t,s)=\frac{-c_f}{1-\beta}\left(1-\frac{a\beta}{\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a}\widetilde{r}_2^{\,s+1-s^*(w_t)}\right)\\
&+w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^s\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\left(1+\frac{-\rho\widetilde{\beta}_\alpha+\rho\beta-\rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r}_2\left(\frac{1}{\beta}-\frac{1}{\widetilde{\beta}_2}\right)+a\right)}\left(\varphi^{\frac{1}{1-\alpha}}\right)\left(\frac{\widetilde{r}_2}{\varphi^{\frac{1}{1-\alpha}}}\right)^{s+1-s^*(w_t)}\right)
\end{aligned}$$

Solving for χ: χ is such that the continuation value at $s = s^*(w_t)$ is equal to zero. The continuation value is

$$aV(w_t, s_t^* - 1) + b + cV(w_t, s_t^* + 1) = \frac{-c_f}{1-\beta}\left(1 - \frac{a\beta}{\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a}\left(a + b\widetilde{r_2} + c\widetilde{r_2}^2\right)\right)$$

$$+ w_t^{\frac{-\alpha}{1-\alpha}}\left(\varphi^{\frac{1}{1-\alpha}}\right)^{s^*(w_t)}\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\left(\rho + \frac{-\rho\widetilde{\beta}_\alpha + \rho\beta - \rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a\right)}\left(a + b\widetilde{r_2} + c\widetilde{r_2}^2\right)\right)$$

$$= \frac{-c_f}{1-\beta}\left(1 - \frac{a\beta}{\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a}\frac{\widetilde{r_2}}{\widetilde{\beta}_2}\right) + \chi\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\left(\rho + \frac{-\rho\widetilde{\beta}_\alpha + \rho\beta - \rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a\right)}\frac{\widetilde{r_2}}{\widetilde{\beta}_2}\right)$$

The last expression is independent of w_t. Thus, to solve for χ we just need to equate the above to zero and this yields

$$\chi = \frac{\frac{c_f}{1-\beta}\left(1 - \frac{a\beta}{\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a}\frac{\widetilde{r_2}}{\widetilde{\beta}_2}\right)}{\frac{(1-\alpha)\alpha^{\frac{\alpha}{1-\alpha}}}{1-\rho\widetilde{\beta}_\alpha}\left(\rho + \frac{-\rho\widetilde{\beta}_\alpha + \rho\beta - \rho\beta a\varphi^{\frac{-1}{1-\alpha}}\widetilde{\beta}_\alpha}{\beta\left(\widetilde{r_2}\left(\frac{1}{\beta} - \frac{1}{\widetilde{\beta}_2}\right) + a\right)}\frac{\widetilde{r_2}}{\widetilde{\beta}_2}\right)}$$

which completes the proof. □

C Data Appendix

In this appendix, we describe the different data sources used in the paper. The first data source is the Business Dynamics Statistics (BDS), giving firm counts by size and age on the universe of firms in the US economy. Compustat data contains information on publicly traded firms. Finally, we use publicly available aggregate time series.

C.1 BDS data

According to the US Census Bureau, the Business Dynamics Statistics (BDS) provides annual measures of firms' dynamics covering the entire economy. It is aggregated into bins by firm characteristics such as size and size by age. The BDS is created from the Longitudinal Business Database (LBD), a US firm-level census. The BDS database gives us the number of firms by employment size categories (1-5, 5-10, 10-20,20-50,50-100,100-250,500-1000,1000-2500,2500-5000,5000-10000) for the period ranging from 1977 to 2012. Note that the number of firms in each bin is the number of firms on March 12 of each year. We also source from the BDS the number of firms of age zero by employment size. We call the latter entrants.

We compute the empirical counterpart of the steady-state stationary distribution in our model based on this data, by taking the average of each bin over years. We do this for the entrant and incumbent distributions. We then estimate the tail of these distribution following Virkar and Clauset (2014). We find that the tail estimate for the (average) incumbent size distribution is 1.0977 with a standard-deviation of 0.0016. For entrants, this estimate is 1.5708 with standard deviation of 0.0166. To compute the entry rate, we divide the average number of entrants over the period 1977-2012 by the average number of incumbents. Over this period there are 48,8140 entrant firms and 4,477,300 incumbent firms; the entry rate is then 10.9%.

To perform the exercise described in Section 5.4, we need to compute the model counterpart of the time t firm size distribution. According to Theorem 1, these are deviations of the firm size distribution around the (deterministic) stationary firm size distribution. However, in the BDS data, the trend of each bin is different. We thus HP-filter each bin of the BDS data with a smoothing parameter $\lambda = 6.25$. Each bins is thus decomposed $\mu_{s,t}^{BDS} = \mu_{s,t}^{BDS-trend} + \mu_{s,t}^{BDS-dev}$ where $\mu_{s,t}^{BDS}$ is the original bin value, $\mu_{s,t}^{BDS-trend}$ is its HP-trend and $\mu_{s,t}^{BDS-dev}$ is the HP-deviation from trend. The empirical counterpart of time t firm size distribution in our model is thus $\mu_{s}^{BDS-average} + \mu_{s,t}^{BDS-dev}$ where $\mu_{s}^{BDS-average}$ is the average of bins s over the period 1977-2012. We then use Equations 1, 3 and 12 to compute the time series for aggregate TFP, Y_t and $\frac{\sigma_t^2}{T^2}$ which we plot in Figure 6 along with data aggregate time series describe below.

C.2 Compustat

The Compustat database is compiled from mandatory public disclosure documents by publicly listed firm in the US. It is a firm-level yearly (unbalanced) panel with balance sheet information. Apart from firm-level identifiers, year and sector (4 digit SIC) information, we use two variables from Compustat: employment and sales. We use data from the year 1958 to 2009. Sales is a nominal variable. We deflate it using the price deflator given by the NBER-CES Manufacturing Industry Database for shipments (PISHIP) in the corresponding SIC industry.

Using this dataset, we estimate tail indexes following Gabaix and Ibragimov (2011), performing a log rank-log size regression on the cross-section of firms each year. Our measure of size is given by the number of employees. We compute tail estimates for firms above 1k, 5k, 10k, 15k and 20k employees. We then HP-filter the resulting time-series of tail estimates (with a smoothing parameter of 6.25).

For each year, we also compute the cross-sectional variance of real sales and then HP-filter the time series using a smoothing parameter of 6.25.

C.3 Aggregate Data

The aggregate data comes from two sources. We take quarterly time series of aggregate TFP and Output from Fernald (2014). For the exercise in Section 5.4, since the BDS data are computed on March 12 of each year, we compute the average over 4 quarters up to, and including, March. For example, for the year 1985 we compute the average of 1984Q2, 1984Q3, 1984Q4 and 1985Q1. We do this for TFP and Output before HP-filtering the resulting time series with a smoothing parameter of 6.25. The other source for annual time series on aggregate output is taken from the St-Louis FED. We use this series for the correlations reported in Table 4, either after HP-filtering with smoothing parameter 6.25 or by computing its growth rate.

For the results in Table 5, we estimate a GARCH(1,1) on the de-meaned growth rate of both aggregate TFP and output, both at a quarterly frequency. The source for this data is Fernald (2014). We take the square of 4 quarter-average of the conditional standard deviation vector resulting from the estimated GARCH. We then HP-filter these series with a smoothing parameter of 6.25.

C.4 Robustness Check

Sample	Firms with more than	1k	5k	10k	15k	20k
Model	Correlation in level	−0.50	−0.71	−0.64	−0.57	−0.48
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
	Correlation in growth rate	−0.11	−0.35	−0.41	−0.42	−0.44
		(0.000)	(0.000)	(0.000)	(0.000)	(0.000)
Data	Correlation in (HP filtered) level	−0.36	−0.17	−0.34	−0.51	−0.46
		(0.005)	(0.20)	(0.008)	(0.000)	(0.000)
	Correlation in growth rate	−0.29	−0.21	−0.33	−0.43	−0.38
		(0.030)	(0.114)	(0.011)	(0.001)	(0.004)

Table 6: Correlation of tail estimate with aggregate output.

NOTE: The tail in the model is estimated for simulated data based on our baseline calibration (cf. Table 2) for an economy simulated during 20,000 periods. The tail in the data is estimated on Compustat data over the period 1958-2008. The aggregate output data is from the St-Louis Fed.

	(1) IQR of Real Sales (Compustat)	(2) STD of Pdy (Durables) (Kehrig 2015)	(3) IQR of real sales (Bloom et al. 2014)
Aggregate Volatility in TFP growth	0.2532	0.3636	0.3583
	(0.0825)	(0.0269)	(0.030)
Aggregate Volatility in GDP growth	0.1911	0.2923	0.3504
	(0.1932)	(0.079)	(0.034)

Table 7: Correlation of Dispersion and Aggregate Volatility

NOTE: In this table, we display the correlation of various measures of micro-level dispersion with two measures of aggregate volatility. Aggregate volatility is measured by the fitted values of an estimated GARCH on growth rates of TFP and output. Both are sourced from Fernald (2014) (see description above). In column (1) the Inter Quartile Range (IQR) of real sales is computed using Compustat data from 1960 to 2008 for manufacturing firms. Nominal values are deflated using the NBER-CES Manufacturing Industry Database 4-digits price index. In column (2) we take the establishment-level median standard deviation of productivity (levels) from Kherig (2015) who, in turn, computes it from Census data. In column (3) we take the establishment-level IQR of sales growth from Bloom at al. (2014).

D Numerical Appendix

In this numerical appendix, we first describe the numerical solution algorithm and its implementation and assess the accuracy of the solution. We then present a set of results obtained under an alternative calibration strategy.

D.1 Solution Method and Accuracy

In this appendix, we describe the numerical algorithm used to solve the model described in the paper. Recall that given the Equation 2, A_t is a sufficient statistic to describe the wage. Using Equation 15, it is clear that the law of motion of A_t is a function of past values of A_t, $E_t(\varphi)$, and σ_t. As described in the main text, we are assuming that firms do not take into account the time-varying volatility of A_t and form their expectations by assuming that $\frac{E_t(\varphi)}{A_t}$, $\frac{O_t^A}{A_t}$ and $\frac{\sigma_t}{A_t}$ are constant and equal to their steady-state value. It follows that, from the perspective of the firms, A_t only depends on its past value.[40]

[40] We also explored the alternative assumption that firms form their expectations by assuming that $E_t(\varphi)$, O_t^A and σ_t are constant and equal to their steady-state value. With this alternative assumption, the policy function is barely affected and all the results in the paper are quantitatively very similar.

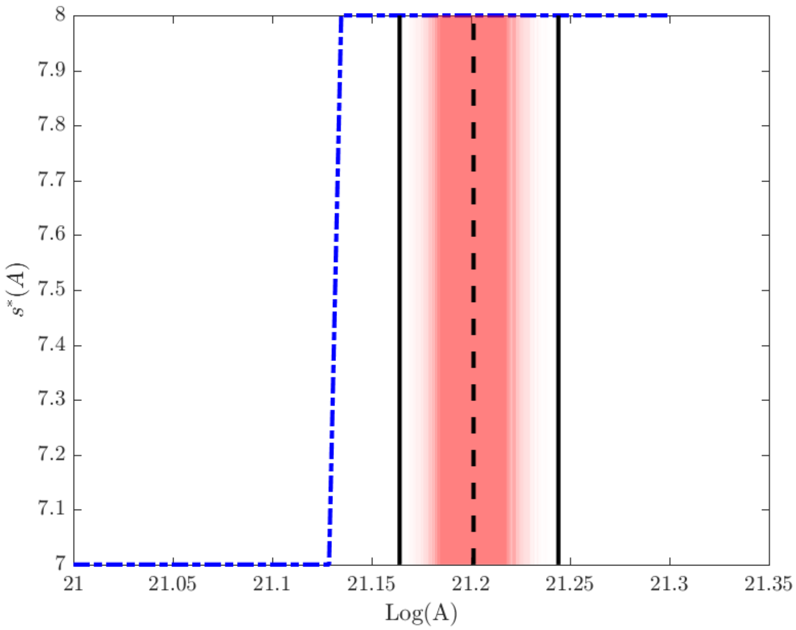

Figure 7: Policy Function and stochastic Domain of A_t

NOTE: The blue (dash-dot) line is the policy function $s^*(A)$; For a 20 000 period simulation of the model, the vertical black (solid) lines are the minimum and maximum of $\log(A_t)$ over this sample; the black (dashed) line is the mean of $\log(A_t)$ over this sample; each of the vertical red (transparent) lines represent $\log(A_t)$ for a given time t.

It follows that the value of being a incumbent only depends on A. To solve the model we simply have to solve for the following Bellman equation:

$$V(A, \varphi^s) = \pi^*(A, \varphi^s) + \max\left\{0, \beta \int_{A'} \sum_{\varphi^{s'} \in \Phi} V(A', \varphi^{s'}) F(\varphi^{s'}|\varphi^s) \Upsilon(dA'|A)\right\}$$

where $\Upsilon(.|A)$ is the conditional distribution of next period's state A', given the current period state A. This distribution is given by Equation 15 with $\frac{E_t(\varphi)}{A_t} = \frac{E(\varphi)}{A}$, $\frac{O_t^A}{A_t} = \frac{O^A}{A}$ and $\frac{\sigma_t}{A_t} = \frac{\sigma}{A}$. We also assume that the shock ε_{t+1} in this last equation follows a standard normal distribution, which is a valid approximation as shown in the Theorem 3 in Appendix B.5.

To solve for the above Bellman equation we are using a standard value function iteration algorithm implemented in Matlab with the Compecon toolbox developed by Miranda and Fackler (2004). To do so, we define a grid for A (in logs) along with productivity grid of the idiosyncratic state space Φ described in the paper. We then form a guess on the value function as a function of $\log(A)$ and $\log(\varphi^s)$, and plug it to the right hand side of the above Bellman equation. This is repeated until convergence. This algorithm converges to the solution of the above Bellman equation and allows us to compute the policy function $s^*(A)$. Figure 7 displays this policy function computed from the value function iteration procedure described above. In this figure, we also plot the ergodic domain of $\log(A_t)$ for a 20 000 period simulation of our model (using the results in Theorem 1). We observe that the value of $\log(A_t)$ is concentrated on the part of the state space where the policy function s^* is constant. Note this is a numerical result rather than an assumption.

Given that firms solve their problem under the perceived law of motion given by Assumption 3, it is important to see if there is an important deviation of this perceived law of motion from the actual law of motion described in Theorem 3. To see this, we plot the two implied aggregate TFP time series for a simulation path of our model in Figure 8. We observe that the actual (blue solid) and the perceived

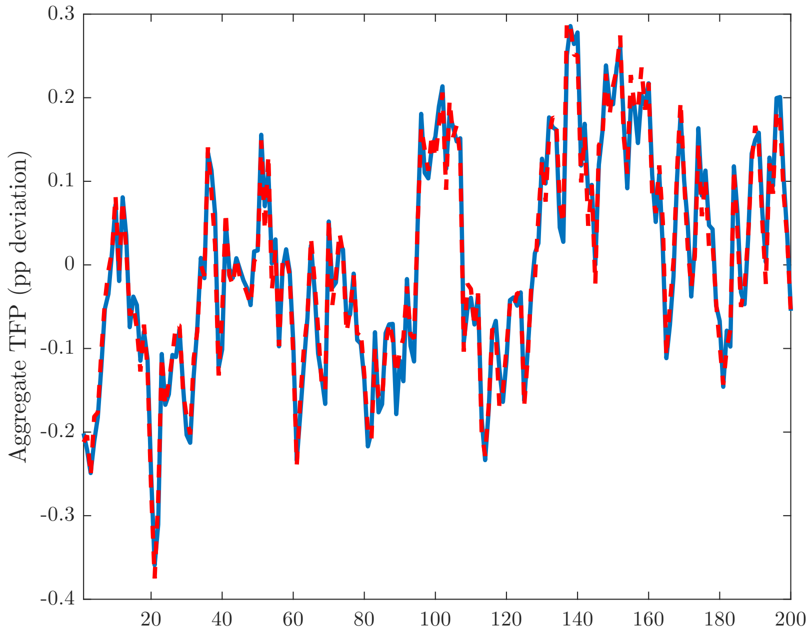

Figure 8: A simulated path for aggregate TFP under the actual and perceived law of motions.
NOTE: The red (dashed) line is the actual times series of aggregate TFP (by Theorem 3); The blue (solid) line is the time series of aggregate TFP implied by Assumption 3. The correlation between these two series is 0.9963.

(red dashed) series follow each other closely. Furthermore, on a 20 000 periods simulated path, the correlation between these two series is 0.9963.

D.2 Alternative Calibration

In this section, we explore an alternative calibration strategy. Instead of fixing the value of α, the span of control parameter, and then matching the idiosyncratic volatility of productivity σ_e, we are now matching the volatility of idiosyncratic sales while fixing the volatility of idiosyncratic productivity in the steady-state. To do so, we calibrate the value of α rather than fix it. For the idiosyncratic volatility of sales, we choose a 35% target following Gabaix (2011) and Comin and Mulani (2006). The targets of this alternative calibration are summarized in Table 8, while the implied parameters can be found in Table 9. Note that the calibrated α is now equal to 0.77. Figure 9 plots the firm size distribution and the entrant distribution in the steady-state of the model against their counterpart in the data.

The results are qualitatively unchanged. If anything, the implied aggregate volatility is stronger as the reallocation mechanism is weaker. We reproduce here the business cycle statistics (Table 10) described in Section 5.2.2, the impulse response to a one standard deviation negative shock on the largest firm (Figure 10) described in Section 5.2.3, and the variation of the counter cumulative distribution function for a simulated path (Figure 11) described in Section 5.3.

Statistic	Model	Data	References
Entry Rate	0.850	0.109	BDS firm data
Sales Vol.	0.35	$0.2 - 0.4$	See main text
Tail index of Firm size dist.	1.097	1.097	BDS firm data
Tail index of Entrant Firm size dist.	1.570	1.570	BDS firm data
Share of Employment of the largest firm	0.17%	1%	Share of Wall-Mart
Number of Firms	4.5×10^6	4.5×10^6	BDS firm data

Table 8: Targets for the calibration of parameters (alternative calibration)

Parameters	Value	Description
a	0.5980	Pr. of moving down
c	0.4020	Pr. of moving up
S	41	Number of productivity levels
φ	1.0868	Step in pdty bins
Φ	$\{\varphi^s\}_{s=1..S}$	Productivity grid
γ	2	Labor Elasticity
α	0.77	Production function
c_f	1.0	Operating cost
c_e	0	Entry cost
β	0.95	Discount rate
M	$3.6435 * 10^7$	Number of potential entrants
G	$\{MK_e(\varphi^s)^{-\delta_e}\}_{s=1..S}$	Entrant's distr. of the signal
K_e	0.7652	Scale parameter of the distr. G
$\delta_e(1-\alpha)$	1.570	Tail parameter of the distr. G

Table 9: Alternative calibration

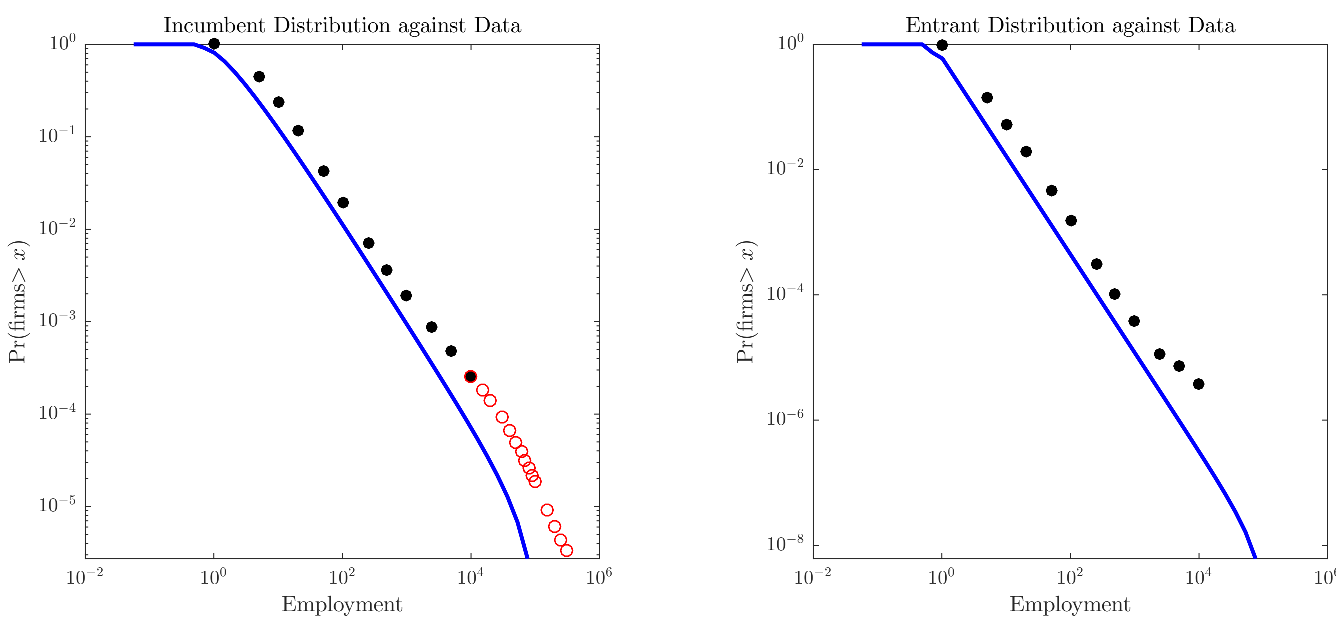

Figure 9: Counter Cumulative Distribution Functions (CCDF) of the firms size distribution of incumbents (left) and entrants (right) in the model (blue solid line) against data (circles).

NOTE: Black filled circle report the CCDF of firm size distribution for less than 10000 employees in the BDS. The red circle are tabulation from Compustat for firms with more than 10000 employees assuming that for this range the distribution of firms in Compustat is similar to the one of firms in the whole economy.

	Model			Data		
	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(y)}$	$\rho(x,y)$	$\sigma(x)$	$\frac{\sigma(x)}{\sigma(y)}$	$\rho(x,y)$
Output	0.54	1.0	1.0	1.83	1.00	1.00
Hours	0.36	0.66	1.0	1.78	0.98	0.90
Aggregate TFP	0.26	0.48	1.0	1.04	0.57	0.66

Table 10: Business Cycle Statistics

NOTE: The model statistics are computed for the alternative calibration (cf. Table 9) for an economy simulated for 20,000 periods. The data statistics are computed from annual data in deviations from an HP trend. The source of the data is Fernald (2014). For further details refer to Appendix C.

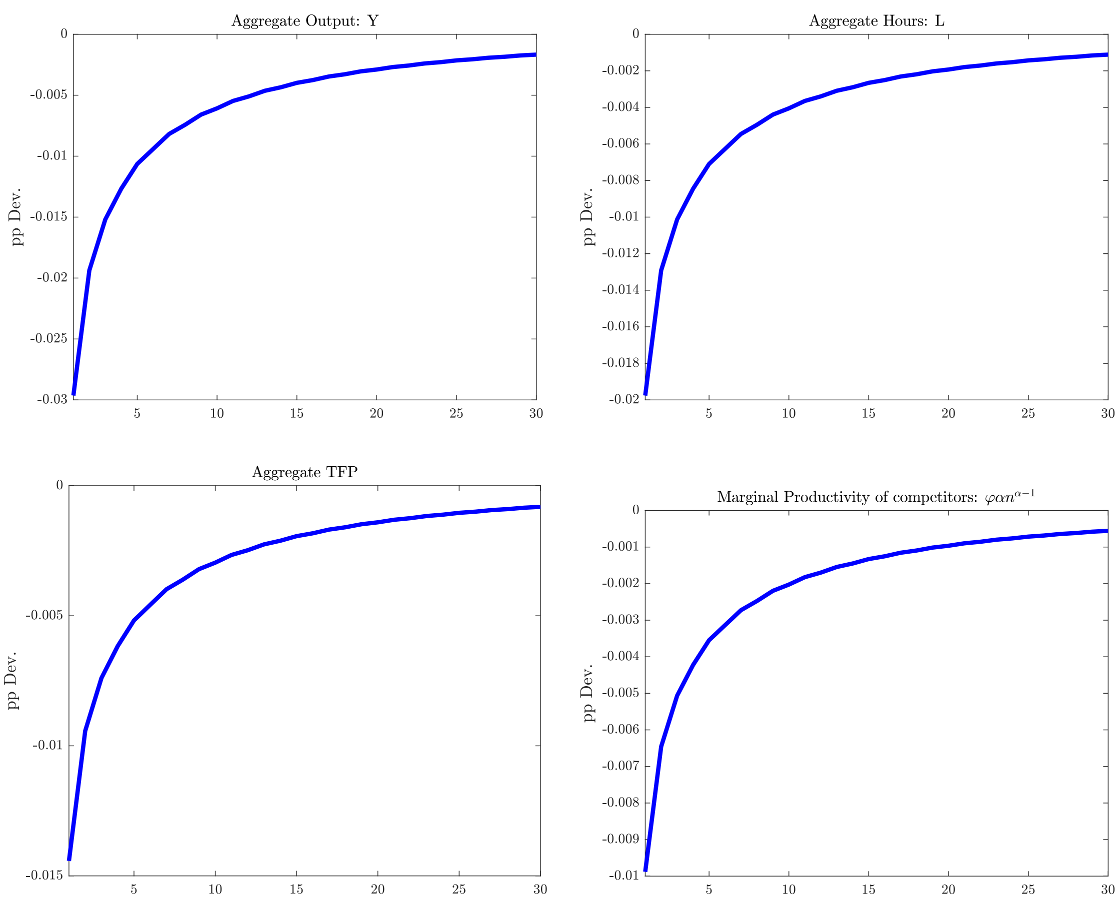

Figure 10: Impulse response to a one standard deviation negative productivity shock on the largest firm.

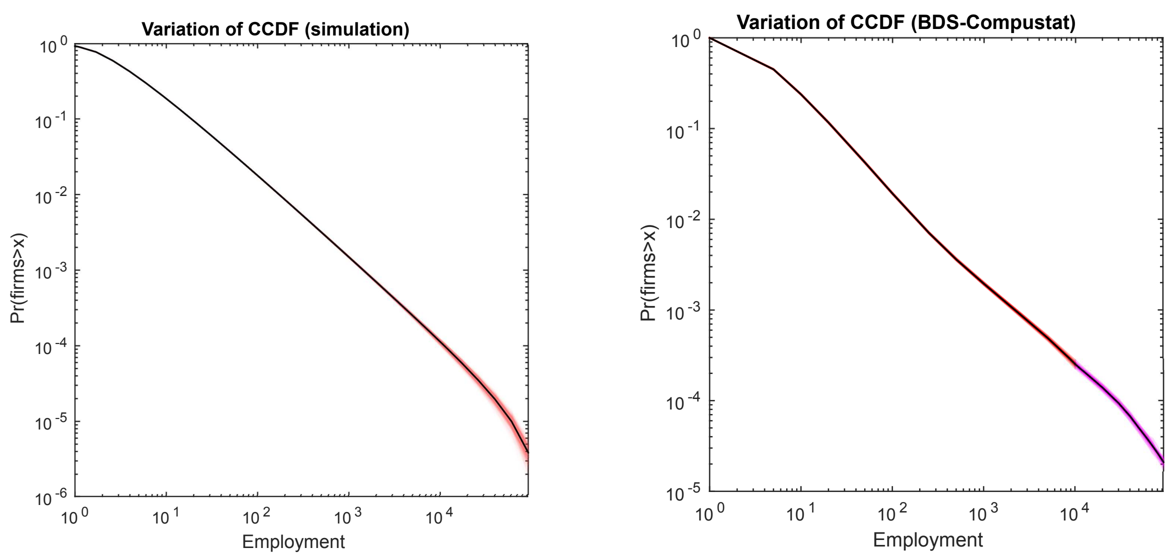

Figure 11: Variation of the Counter Cumulative Distribution Function (CCDF) in simulated data (left) and in the BDS data (right).

NOTE: The simulated data are the results of a 25000 periods sample (where the first 5000 are dropped). For the BDS data, we compute the CCDF for each year on the sample 1977-2008. The dashed black line is the mean of each sample.