Aggregate Fluctuations and the Network Structure of Intersectoral Trade

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Abstract

This paper analyzes the flow of intermediate inputs across sectors by adopting a network perspective on sectoral interactions. I apply these tools to show how fluctuations in aggregate economic activity can be obtained from independent shocks to individual sectors. First, I characterize the network structure of input trade in the U.S.. On the demand side, a typical sector relies on a small number of key inputs and sectors are homogeneous in this respect. However, in their role as input-suppliers sectors do differ: many specialized input suppliers coexist alongside general purpose sectors functioning as hubs to the economy. I then develop a model of intersectoral linkages that can reproduce these connectivity features. In a standard multisector setup, I use this model to provide analytical expressions linking aggregate volatility to the network structure of input trade. I show that the presence of sectoral hubs - by coupling production decisions across sectors - leads to fluctuations in aggregates.

Keywords: Aggregation; Business Cycles; Comovement; Input-Output; Multisector Growth Models; Networks; Technological Diversification

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1 Introduction

Comovement across sectors is a hallmark of cyclical fluctuations. A long-standing line of research in the business cycle literature asks whether trade in intermediate inputs can link otherwise independent technologies and generate such behavior. The intuition behind this hypothesis is clear: factor demand linkages can provide a source for comovement, as a shock to the production technology of a general purpose sector - say, petroleum refineries - is likely to propagate to the rest of the economy. In this way, cyclical fluctuations in aggregates are obtained as synchronized responses to changes in the productivity of narrowly defined but broadly used technologies.

Though intuitive, this hypothesis is faced with a strong challenge: by a standard diversification argument, as we disaggregate the economy into many sectors, independent sectoral disturbances will tend to average out, leaving aggregates unchanged and yielding a weak propagation mechanism; see the discussion in Lucas (1981) and the irrelevance theorems of Dupor (1999)[1].

In this paper, I take on this challenge by adopting a network perspective on sectoral interactions. From this vantage point, I provide answers to the following questions. First, given the availability of detailed input use data, can we identify the main features of the network structure of linkages across sectors? Second, can we specify models of sectoral input linkages that are able to mimic this connectivity structure and are still amenable to use in standard multi-sector models? If so, can we use these models to derived analytical results linking the variability of aggregates to the network structure of input flows? Finally, under what assumptions on the network structure can we render ineffective the shock diversification argument of the previous paragraph?

The argument linking the answers to these questions is the following: when determining whether a sectoral shock propagates or not, the number of sectoral connections originating from the source of the shock is the crucial variable to consider. Furthermore, if the number of

[1]For the most part, the answer to this challenge has been in the empirical vein. Long and Plosser (1983, 1987), Norrbin and Schlagenhauf (1990) and Horvath and Verbrugge (1996) document comovement of sectoral output growth series through vector autoregressions. They all add that the explanatory power of a common, aggregate shock is limited on its own and diminished once sector specific shocks are entertained. Shea (2002) and Conley and Dupor (2003) go further and devise ways of testing - and rejecting- the hypothesis that sectoral comovement is being driven by a common shock. Concurrently, the strategy of using actual input-output data in large scale multisector models generates aggregates that are quantitatively similar to data and to one-sector real business cycle models; see Horvath (2000).

connections varies widely across sectors, some shocks will propagate throughout the economy and persist through time while others will be short-lived and only propagate locally. As a consequence, economies where every sector relies heavily on only a few sectoral hubs - general purpose input suppliers - will show considerable conductance to shocks in those technologies. Conversely, as the structure of the economy is more diversified, different sectors will rely on different technologies and exhibit only loosely coupled dynamics. The answer to the law of large numbers arguments in Lucas and Dupor thus lies in understanding and modelling this tension between specialization and reliance on general purpose technologies.

A simple example - a particular case of the setup in Shea (2002)- helps build intuition for this tension[2]. Consider an economy where a representative household derives log utility over M sectoral goods and linear disutility from time spent working. Each of the M sectors produce a different good that can either be allocated to final consumption or as an intermediate input to the production of other goods. In particular, let the production side of this economy be given by M, Cobb-Douglas, constant returns to scale production functions, each combining labor and a distinct set of intermediate inputs. Finally, assume that each sector is subject to a productivity shock of variance σ^2 but insist that these shocks are *independent* realizations across sectors.

Whether these shocks will then propagate through input linkages and lead to movements in aggregates depends on the network structure of these linkages. To see this, consider the two following abstract, and rather extreme, cases. Fix an M and contrast an economy where only one sector is a material input supplier to all other sectors with an economy where every sector supplies to all other sectors in the economy. These two polar cases for the pattern of input-use relationships in an economy map exactly into very standard network representations, where the vertex set is given by the set of sectors in the economy and a directed arc from vertex (sector) i to vertex j represents a intermediate input supply link.

[2] This setup, along with all the results pertaining to it, is carefully spelled out in Section 3 of the paper.

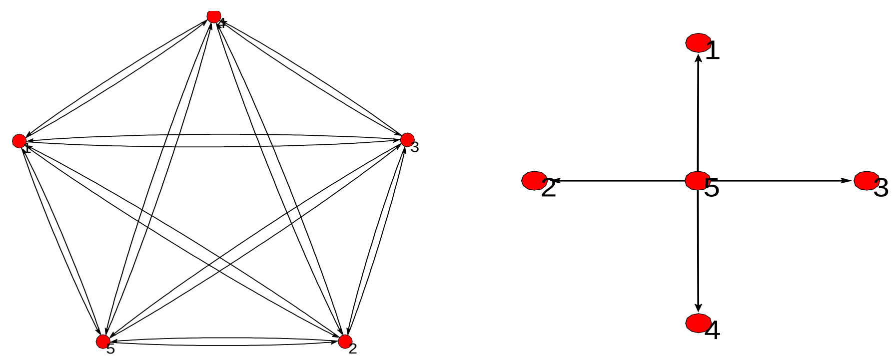

Figure 1: *Complete (l.h.s.) and Star (r.h.s.) input-supply structures for a 5 sector economy.*

Thus an economy where each sector is an input supplier to every other sector in the economy can be represented by a *complete* network, where for any two pair of vertices there is a directed arc from one to the other. Likewise, an economy where there is only one material input supplier maps directly into a *star* network, where one vertex acts as a hub with directed arcs from this vertex to all other vertices. An intermediate case is given by a $N-$*star* network, where N out of M, sectors in the economy act as material input suppliers to every sector and the remaining ones are solely devoted to final goods production. Figure 1 depicts intersectoral input relations under these two extreme cases - complete and star - for a five sector economy.

In order to focus on the impact of heterogeneity along the extensive margin of intersectoral trade, assume further that, each sector, regardless of what its particular input list is, uses its inputs in equal proportions. Then, defining aggregate volatility, σ_Y^2, as the variance of average log output, I show that $\sigma_Y^2 \propto \frac{\sigma^2}{M}$ for the case of complete networks while $\sigma_Y^2 \propto \frac{\sigma^2}{N}$ for the case of $N-$star networks.

Thus, aggregate volatility with complete intersectoral networks echoes Dupor's and Lucas' law of large numbers argument: aggregate volatility scales with $1/M$. To understand how effective the shock diversification argument is notice the following: holding sectoral variance fixed as I move from a five sector economy to a five hundred sector economy, aggregate volatility will be a hundred times smaller. Conversely, to recover an aggregate σ_Y^2 of the order of two percent in a five hundred sector economy, would require stipulating sectoral volatilities, σ^2, to be five hundred times larger, an unreasonable magnitude at any time scale.

However, if there are only N sectors acting as intermediate input suppliers, the diversification of shocks argument underlying law of large number arguments only applies to those sectors. Thus, in an economy where the effective number of input suppliers is small, the law of large numbers will be postponed relative to that of Dupor (1999): aggregate volatility now scales with $1/N$. This is Horvath's (1998) argument: limited sectoral interaction - of a very particular form - will give rise to greater aggregate volatility from sector specific shocks. The difficulty with this result is that the modeler is now left to specify, for each M, what is the number of input suppliers in an economy, N. From input-output data, Horvath (1998) argues that N - the number sectors with full rows in input-output matrices - grows slowly with M: Horvath argues for an N of order $\sqrt{M}$. This would now yield a ten fold decrease in aggregate variability as we move from five to five hundred sectors.

In this way, two very particular assumptions on the network structure of intersectoral trade generate predictions on the variability of aggregates that differ by an order of magnitude. This means that finding a better way to model networks of input trade can not only help solve this controversy but also has the potential of offering a theory where reasonable magnitudes of sectoral volatility yield non-trivial aggregate volatility. Mechanically, we need only a theory of intersectoral connectivity that yields aggregate volatility decaying with M^{-v}, where v is close to zero. This paper does just this by going beyond these two extreme cases and building a model of sectoral interactions on a network. Figure 2 depicts the starting point of the analysis. It shows a considerably more intricate network of intersectoral input flows: that of the U.S. economy in 1997.

Each dot - or vertex - corresponds to a sector defined at the NAICS 4-6 digit level of disaggregation in the BEA detailed commodity-by-commodity tables, for a total of 474 sectors. Each link in the figure represents an input transaction between sector i to sector j, provided sector i supplies more than 5% of sector j total intermediate input purchases[3].

[3] I exclude loops from the network for presentation purposes. Loops correspond to *intra*sectoral trade and are a well documented feature of detailed input use-matrices (see for example Jones, 2010b).

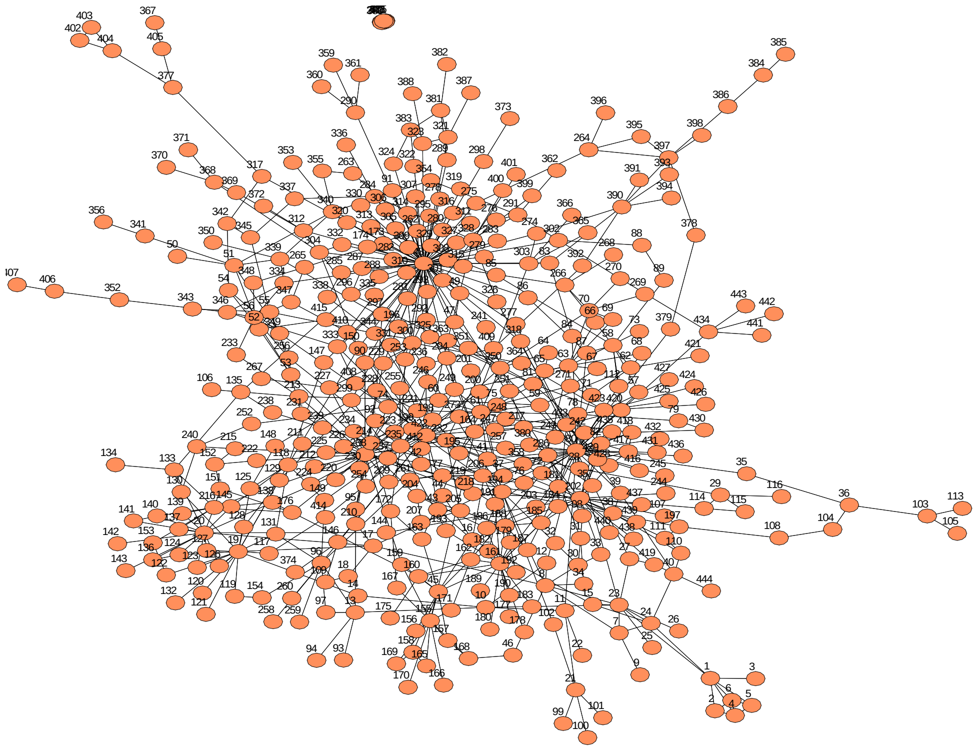

Figure 2: *Intermediate input flows between sectors in the U.S. economy in 1997. Each vertex corresponds to a sector in the 1997 benchmark detailed commodity-by-commodity direct requirements matrix (Source: BEA). For every input transaction above 5% of the total input purchases of the destination sector, a link between two vertices is drawn.*

From this vantage point, Section 2 in the paper offers a two-pronged characterization of the structure of input flow data by taking into consideration the direction in each of these links. Thus, by considering links from the perspective of the destination vertex, I can analyze sectors in their role as input-demanders. I find that sectors are homogeneous along this dimension: the typical sectoral production technology relies on a relatively small number of key inputs and sectors do not differ much in this respect. This is the upshot of specialization occurring at the level of narrowly defined production technologies.

However, looking at the source vertices of these links, another feature emerges: extensive heterogeneity across sectors in their role as input suppliers. In the data, highly specialized input suppliers - say, for example, optical lens manufacturing - coexist alongside general purpose inputs, such as iron and steel mills or petroleum refineries. Specifically, I characterize the empirical out-degree distribution of input-supply links - giving the number of sectors to which any given sector supplies inputs to - as a power law distribution. What makes this power law parameterization attractive is the following argument: the upshot of fat-tails, characteristic of power law degree distributions, is that a small, but non-vanishing, number of sectors will emerge as large input suppliers - or hubs - to the economy.

In Section 3, I construct a network model of intersectoral linkages that is able to incorporate these two first order-features of the data: sparse and homogeneous input demand and strongly heterogenous input supply technologies. Returning to the static multisector setup described above, I then show that whenever the network of input linkages incorporate these two features convergence of aggregate volatility to zero can be slowed down dramatically. In particular, I show that in this setup $\sigma_Y^2 \propto \sigma^2 \left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}$ where $\zeta \in (2,3)$ is the tail parameter on the power law distribution of input-supply linkages. Given the empirical characterization of Section 2, $\zeta = 2.1$ seems to be a good description of actual input-use data. Thus, $\sigma_Y^2 \propto \sigma^2 \left(\frac{1}{M}\right)^{0.18}$. This means that going from a five to a five-hundred sector economy - while keeping sector-level volatility constant - now implies that aggregate volatility is reduced only *two-fold*. Alternatively, we need only that the typical sectoral volatility of narrowly defined sectors be double than that of more aggregated sectors for aggregate volatility to remain constant across these two economies.

The remainder of the paper (Section 4) is devoted to verifying that these claims still hold in standard, dynamic, multisector setups. In particular, I show that the decay characterization above extends to the auto-covariance function of aggregate output growth. I then present some quantitative explorations in this class of models and claim that the mechanisms described in this paper are quantitatively relevant: a large scale multisector model with independent shocks can generate aggregate volatility that is about two-thirds of that observed in data.

The paper is closest to the contribution of Gabaix (2010) and to the independent, but subsequent, work by Acemoglu, Ozdaglar and Tahbaz-Saleh (2010). Regarding Gabaix (2010), this paper is closely related to his characterization of aggregate volatility decay as a function of heterogeneity in the underlying production units. In contrast to Gabaix however, this is not the result of some firms accounting for a non-trivial share of aggregate output and thus, for a non-trivial share of aggregate volatility. Rather the argument here is based on the shock conductance implied by the interlocking of technologies in a networked economy. In other words, the emphasis here is on propagation rather than aggregation. These two approaches should therefore be seen as a complementary. Acemoglu et al (2010), consider a networked, static, multisector economy which is very similar to the one discussed earlier in this introduction and expanded further upon in Section 3. They use a different approach (deterministic graphs, rather than the random graphs setup used here) which allows them to confirm the volatility decay behavior discussed above - stated in Proposition 2 of this paper- and then analyze higher-order network effects and the possibility of tail events in the aggregate. They do not study whether this is still the case in dynamic settings which allow for richer interactions due to the presence of capital accumulation - as is done here- nor do they look at the quantitative implications of these settings.

The underlying multi-sector setup that I use is very close to that appearing in Horvath (1998), Dupor (1999), Shea (2002) and Foerster, Sarte and Watson (2008), all closely related to the original multisector real business cycle model of Long and Plosser (1983) and the myriad of extensions and applications developed in the literature since. Much of this literature has used actual input-output data in the calibration of more complicated large-scale multisector equilibrium models with some quantitative success; see Horvath (2000), Kim and Kim (2006) or Bouakez, Cardia and Ruge-Murcia (2009) for examples of this. This papers asks what in the nature of input-output data is enabling such results, unexpected in a context of independent sectoral shocks.

The paper is also close to the spirit of the contributions in Bak et al.(1993) and Scheinkman and Woodford (1994) by stressing the importance of the structure of input-supply chains in the transmission of shocks across sectors and, as a consequence, to aggregates. In comparison with these papers, by placing sectors on a network of input flows - rather than on a lattice - I allow for more general, and arguably more realistic, patterns of connections between sectors[4]. The idea of characterizing input-use relationships through graph-theoretical tools is not new, albeit it has merited only limited attention[5]. In the context of traditional input-output analysis Solow (1952) is, to the best of my knowledge, the first reference recognizing that an input-output matrix can be mapped into a network. These tools have resurfaced only sporadically in the analysis of static and dynamic input-output systems; see Rosenblatt (1957), Simon and Ando (1961) or Szydl (1985).

In terms of tools, this paper borrows heavily from recent work on networks and in particular, random graphs. Newman (2003) and Li et al (2006) offer good reviews mapping out recent theoretical advances and link them to a growing number of applications. Durrett (2006) and Chung and Lu (2006) provide textbook treatments. In particular, a model of

[4]Other related setups have been explored in order to generate aggregate fluctuations from micro shocks: Cooper and Haltiwanger (1990). Jovanovic (1987) and Durlauf (1993) instead focus on the role of production complementarities across sectors, as does the more recent contribution of Nirei (2005), where this is coupled with indivisibilities in investment. In turn, Murphy et al. (1989) focus on aggregate demand spillovers.

[5]This stands in sharp contrast to the recent but burgeoning use of network tools in microeconomics; see Jackson (2005) for a comprehensive review or Vega-Redondo (2007), Jackson (2008) or Goyal (2009) for monographs on the topic.

random graphs with given expected degree sequences, set out in Chung and Lu (2006), forms the basis for my data-generating process for intersectoral linkages.

2 Network Properties of Input Flow Data

This section conducts an empirical analysis of some network properties of input flow data. Throughout, I use detailed Benchmark Input-Output data compiled by the Bureau of Economic Analysis, spanning the period 1972-2002. The detailed input-output data yield a fine disaggregation of inter-sectoral trade, most sectors corresponding to (roughly) a four digit S.I.C. definition. The data is made available on a five year interval.[6]

In particular, I use the commodity-by-commodity direct requirements tables[7] where the typical (i, j) entry gives the input-share (evaluated at current producers' prices) of (row) commodity i as an intermediate input in the production of (column) commodity j. Abusing notation slightly, I use the names commodities and sectors interchangeably throughout the paper (i.e. I assume that commodity i is produced exclusively by sector i).

I now map this intersectoral input trade data into standard graph theoretical notation. First, let the set of M sectors in an economy give the set of fixed labels for the vertex set $V \doteq \{v_1, ..., v_M\}$. Let E be a subset of the collection of all ordered pairs of vertices $\{v_i, v_j\}$, with $v_i, v_j \in V$. Define E by:

$$\{\ \{v_i, v_j\} \in V^2 : \{v_i, v_j\} \in E \text{ if } \textit{Sector } i \textit{ supplies Sector } j\}$$

That is, the edge set E, is given by an adjacency relation, $v_i \to v_j$ between elements of the set of all sectors where I allow reflexivity (a sector can be an input supplier of itself). With the collection V of sectors and input supply relations E, I define sectoral trade linkages as a directed graph G :

Definition 1 $G = (V, E)$. *G is a directed sectoral linkages graph with vertex set V and edge set E where each element of E is a directed arc from element i to j .*

[6] Up until 1992, it is based on an evolving a S.I.C. classification whereas the NAICS system was adopted from 1997 on. See Lawson et al. (2002) for a comparison of the two classification systems and in-depth discussion of the data. While individual sectors are not immediately comparable between S.I.C. and N.A.I.C.S., the network structure of these matrices will be shown to be remarkably stable across classification systems.

[7] These are not available for all benchmark years but are possible to construct using the available Use and Make tables following the indications in Shea (1991) or the BEA's own Input-Output manual (Horowitz and Planting (2006)).

A useful representation of a graph is its adjacency matrix, indicating which of the vertices are linked (adjacent). This will be a key object in the sections below and is defined by:

Definition 2 *For a directed sectoral linkages graph* $G(V, E)$ *define the adjacency matrix* $A(G)$ *to be an* $M \times M$ *matrix. If* G *is a directed graph define the* a_{ij} *element of* $A(G)$ *to be* 1 *if there is a directed edge from sector* i *to sector* j *(i.e. if sector* i *is a material input supplier of* j*) and zero otherwise.*

I now characterize the extent of heterogeneity along the extensive margins of input demand and input supply. In particular, I consider the number of different inputs a sector *demands* in order to produce- as measured by the columns sums of the adjacency matrix $A(G)$ - and the number of different sectors a sector *supplies* inputs to - as measured by the row sums of $A(G)$. These count measures can be mapped directly in two graphical objects, namely the indegree and outdegree sequences of an intersectoral graph G.

Definition 3 *The in-degree* d_i^{in} *of a vertex* $v_i \in V$ *is given by the cardinality of the set* $\{v_j : v_j \rightarrow v_i\}$. *The in-degree sequence of a graph* $G(V, E)$ *is given by* $\{d_1^{in}, ..., d_M^{in}\}$.

Figure 3 below, displays the empirical density of sectoral indegrees for every detailed matrix available since 1972. I define the indegree of a sector i as the number of distinct input-demand transactions that exceed 1% of the total input purchases of that sector. By only counting as links input transactions above 1% of a sector's total purchases, I am discarding very small transactions between sectors and focusing on the main components of the bill of goods necessary to the production of any given sector. Indeed, following this threshold rule, I account for about 80% of the total value of intermediate input trade in the US economy in 2002. A similar number obtains for all the other years considered[8].

The demand side picture that emerges from Figure 3 is the following: the average sector in the US economy procures a non-trivial amount of inputs from only a small number of sectors ($\simeq 15$) and sectors do not differ much along this demand margin. In other words, the average indegree is small relative to the total number of sectors and most sectors have an indegree that is close to the average indegree.

[8] Though arbitrary, this counting convention seems necessary as there is no way of distinguishing between, say, an input transaction from sector i to j in the order 10 million dollars and an input transaction from sector k to j two orders of magnitude above. Both get counted as one demand link of sector j. In the appendix, I show that the characterization presented here holds for alternative thresholds.

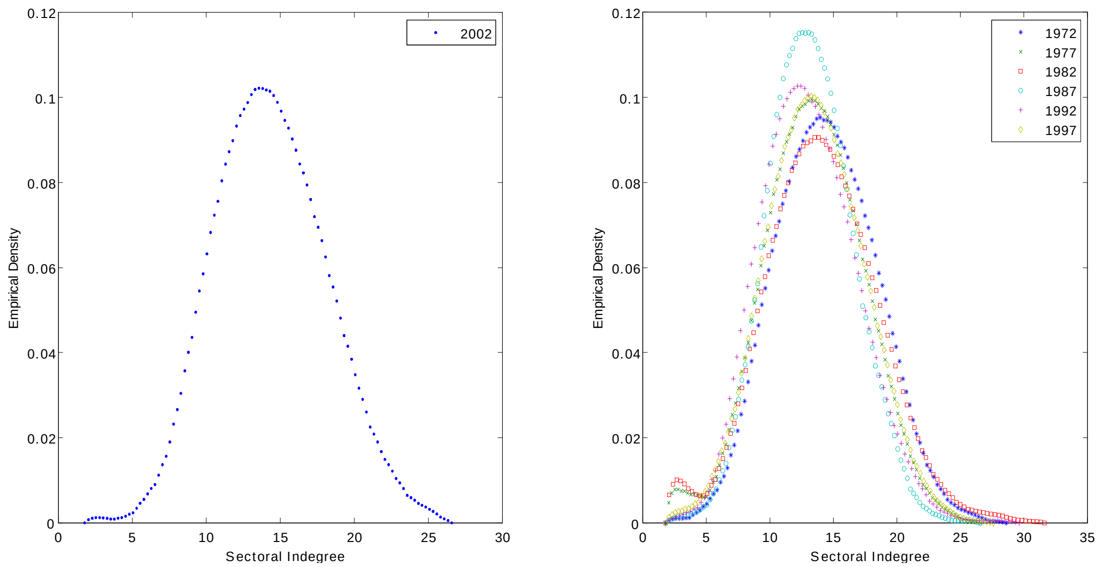

Figure 3: *Empirical density of sectoral indegrees. Only input demand transactions above 1% of the demanding sector's total input purchases are counted. On the l.h.s. is the indegree density for the 2002 detailed direct requirements IO matrix; on the r.h.s. are the empirical densities for direct requirements matrices from 1972 through 1997. Source: B.E.A..*

This can be seen as a way to encode a first-order characteristic of detailed input-output data already alluded to by Horvath (1998) and Jones (2010a and 2010b): these are sparse matrices reflecting specialization occurring at the level of narrowly defined production technologies. Henceforth I'll dub this feature as homogeneity along the extensive margin of sectoral demand. This is to be contrasted with the extreme heterogeneity found along the supply side to which I now turn.

Definition 4 *The out-degree d_i^{out} of a vertex $v_i \in V$ is given by the cardinality of the set $\{v_j : v_i \rightarrow v_j\}$. The out-degree sequence of a graph $G(V, E)$ is given by $\{d_1^{out}, ..., d_M^{out}\}$.*

Figure 4 documents the heterogeneity in sectoral supply linkages by plotting the empirical out-degree distribution in the input-use data where again I use the 1% threshold to define a link. It gives a log-log rank-size plot, i.e. a log-log plot empirical counter-cumulative distribution of the outdegrees, or the probability, $P(k)$, that a randomly selected sector supplies inputs to k or more sectors[9].

[9]The construction of these plots is standard: first, rank all sectors according to the total number of sectors they supply inputs to. Now plot the log of the out-degree of each sector (in the x-axis) against its log rank (in the y-axis). To interpret the plot it is useful to notice the following: if I rank sectors then, by

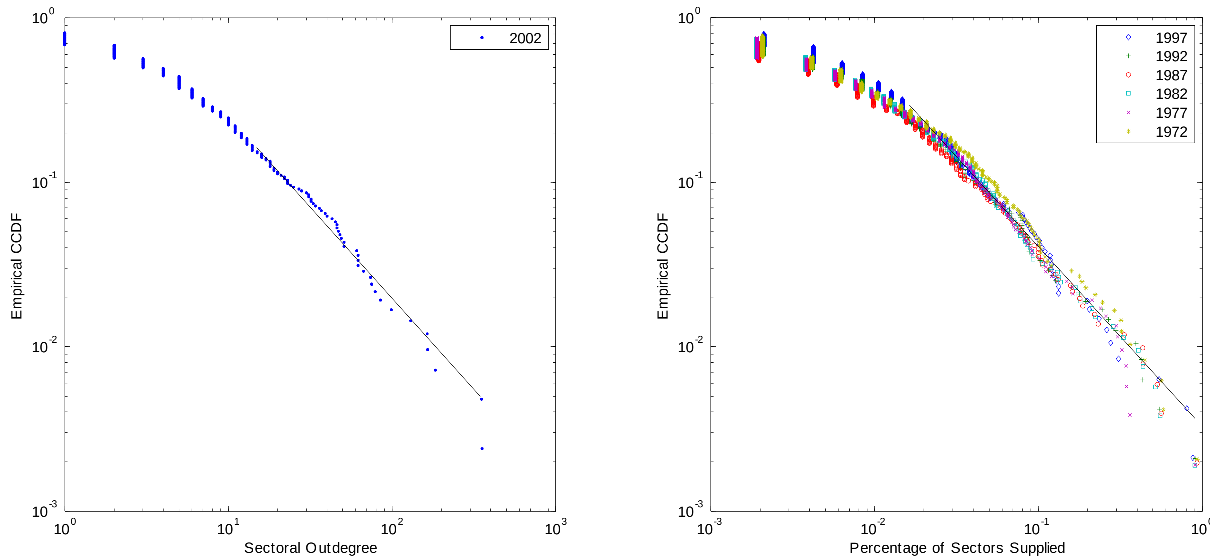

Figure 4: *Counter-cumulative outdegree distribution from direct requirements detailed tables. Only input demand transactions above 1% of the demanding sector's total input purchases are counted. On l.h.s is the 2002 data. The r.h.s. displays 1972 through 1997 data where I normalize the sectoral outdegree d_i^{out} by the total number of sectors in each year. Source: B.E.A.*

Given that every matrix, from 1972 through 1997, differs slightly in its dimensions (i.e. in the number of sectors considered), for every year through 1997, I normalize sectoral outdegrees by the total number of sectors in the input-use matrix. This enables me to compare features of the distributions across different input-use matrices by standardizing the x-axis in the r.h.s of Figure 4.

The apparent linearity in the tail of the (countercumulative) outdegree distribution in log scales is usually associated with a power law distribution[10]. To see this formally, let $P(k) = \sum_{k'=k}^{M} p_{k'}$ be the countercumulative distribution of outdegrees, i.e. the probability that a sector selected at random from the population supplies to k or more sectors. The number of sectors supplied (i.e. the outdegree), k, follows a power law distribution if, the p.d.f. p_k (giving the frequency of sectors that supply to exactly k sectors in the economy) is

definition, there are i sectors that supply inputs to a number of sectors that is greater or equal than that of the $i^{th}-$largest sector. Thus dividing the sector's rank i by the total number of sectors (M) gives the fraction of sectors larger than i.

[10] Which is also a typical feature of the firm size distribution (see, for example, Axtell (2001), Luttmer (2007) or Gabaix (2010)).

given by:

$$p_k = ck^{-\zeta} \text{ for } \zeta > 1, \text{ and } k \text{ integer, } k \geq 1$$

where c is a positive constant and ζ is the tail index. Well-known properties of this distribution are that for $2 \leq \zeta < 3$, k has diverging second (and above) moments[11] while for $1 < \zeta < 2$, k will have diverging mean as well. An estimate on the value of the tail parameter, ζ, can in principle be obtained by running a simple least squares regression of the empirical log-CCDF on the log-outdegree sequence (or its normalized counterpart). However, Clauset, Shalizi and Newman (2009) show that least squares methods can produce substantially inaccurate estimates of parameters for a power-law distribution. Hence, I follow Clauset et al (2009) in implementing Hill-type MLE estimates of $\widehat{\zeta}$ for the tail of the distribution (i.e. using all observations on or above some endogenously determined minimum degree) obtained for every year. I also report the corresponding standard errors and the number of observations in the tail[12].

	1972	1977	1982	1987	1992	1997	2002
$\widehat{\zeta}$	2.22	2.31	2.30	2.03	2.01	2.17	2.05
$s.e.(\widehat{\zeta})$	0.22	0.21	0.22	0.16	0.14	0.17	0.13
N_tail	72	81	68	132	121	93	112
M	483	523	527	509	478	474	417

Table 1: *MLE estimates $\widehat{\zeta}$, their standard errors ($s.e.(\widehat{\zeta})$), the number of observations used to estimate the tail parameter (N_tail) and the total number of sectors (M) for each year from 1972-2002*

The straight lines in Figure 5 show the MLE fit implied by $\widehat{\zeta} = 2.1$ (the average estimate across years is 2.14). From the discussion above, this value of the tail parameter implies a strong fat tailed behavior where the variance is diverging with the number of sectors. This can be taken as a parametric characterization of another feature of input-use matrices already

[11] Though in any finite sample a finite variance can be computed, what this means is that the variance diverges to $+\infty$ as the total number of sectors grows larger. (see Newman, 2003 and 2005, Li et. al., 2006 and Gabaix 2009, for useful reviews and references therein)

[12] Simple OLS estimates of ζ or their modified version -as proposed in Gabaix and Ibragimov (2009) - with an exogenously determined number of sectors on the tail (set at 20% of the number of sectors) lead to very similar point estimates. The appendix shows that when different cutoff rules are used to define a link, similar numbers obtain. See for, example, Brock (1999), Mitzenmacher (2003) or Durlauf (2005) for further discussions on the difficulty of identifying power laws in data.

remarked in Horvath (1998) and Jones (2010a and 2010b): as we disaggregate the economy into finer definition of sectoral technologies, large input-supplying sectors do not vanish. In other words, at the most disaggregated level of sectoral input trade, the distribution of input-supply links is fat tailed.

3 Modelling Networked Sectoral Linkages

When modelling production in a multi-sector context, explicitly accounting for the flows of inputs across sectors entails specifying both a list of intermediate inputs needed for the production of any given sector and the intensity of use of each particular intermediate input in that list. In the particular setting where gross output production functions are Cobb-Douglas, this means specifying the cost shares of intermediate inputs and setting to zero these parameters when a particular input is not required for the production of a given good. According to the analysis of the previous section, one can characterize the zero patterns of these lists as restrictions on the network structure of linkages. This section first shows how to incorporate the two first order network features isolated above in the simplest multisector setup possible. I then show, analytically, how aggregate volatility depends on the network structure of intersectoral linkages.

3.1 A Static Multisector Economy

Consider the following static multisector economy, a particular case of the setup presented in Shea (2002). There is a representative household whose utility is affected by the levels of consumption of M goods, $\{C_j\}_{j=1}^{M}$, and total hours of work , L,to be shared among the M production activities. Assume log preferences over M different goods, with weights given by $\{\theta_j\}_{j=1}^{M}$, and linear disutility of labor.

$$\begin{aligned} U(\{C_j, L_j\}_{j=1}^{M}) &= \sum_{j=1}^{M} \theta_j \log(C_j) - L, & (1) \\ \text{with } \sum_j \theta_j &= 1 \text{ and } \theta_j > 0, \forall_j, & (2) \\ \text{and } \sum_j L_j &\leq L & (3) \end{aligned}$$

Each of the M productive units, or sectors, produce a different good that can either be allocated to final consumption (by the household) or as intermediate goods to be used in the production of other goods. This is just a static version of the production technologies introduced in Long Plosser (1983). In particular, assume production functions are of the Cobb-Douglas, constant returns to scale variety:

$$\begin{aligned} Y_j &= Z_j L_j^{\beta_j} \prod_{i \in \check{S}_j} M_{ij}^{\gamma_{ij}} && (4) \\ 1 &= \beta_j + \sum_{i \in \check{S}_j} \gamma_{ij}, \beta_j > 0, j = 1, ..., M && (5) \\ Z_j &= \exp(\varepsilon_j),\ \varepsilon_j \sim N(0, \sigma_j^2) && (6) \end{aligned}$$

where M_{ij} is the amount of good i used as an intermediate input in the production of sector j. Z_j is a Hicks-neutral, log-normal, productivity shock to good j technology, to be drawn independently across sectors. The 'supply-to' set $\check{S}_j$ completes the description of technology in this simple economy. It gives, for every sector j, the list of goods that are necessary as inputs in the production of good i. Finally, market clearing implies that:

$$Y_j = C_j + \sum_{i:j \in \check{S}_i} M_{ji},\ \ j = 1, ..., M \tag{7}$$

It is a standard exercise to solve for the competitive equilibrium of this economy; see Shea (2002). Substituting the equilibrium input choices into the production function, simplifying and taking logarithms yields, in vector notation:

$$\mathbf{y} = \boldsymbol{\mu} + (I - \Gamma)^{-1'} \boldsymbol{\varepsilon} \tag{8}$$

where $\boldsymbol{\mu}$ is an M-dimensional vector of constants dependent on model parameters only[13]. The pair of vectors M-dimensional vectors $(\mathbf{y}, \boldsymbol{\varepsilon})$ give, respectively, the log of equilibrium output and the log of the productivity shock for every sector in the economy. I is the $M \times M$ identity matrix and Γ is an $M \times M$ input-use matrix with typical element $\gamma_{ij} \geq 0$. The j^{th} column sum of Γ gives the cost share of intermediate inputs for sector j:

$$\gamma_j = \sum_{i=1}^{M} \gamma_{ij}$$

[13]The setup in Shea (2002) also considers preference shocks by making the preference weights on each good stochastic. Shea then shows that the competitive equilibrium solution will be given by (*8*) plus an additional random vector in the right hand side of expression reflecting demand shocks propagating through the input output matrix. In the simplified setup of the current paper this shocks are not present and therefore this a zero vector.

where $\gamma_j < 1$ for all j, such that the $M \times M$ matrix $(I - \Gamma)^{-1}$ is well defined[14]. Thus, in this simple setup, independent technological shocks at the sectoral level propagate through the input-use matrix downstream[15], affecting the costs of input-using sectors and potentially influencing aggregate activity. Henceforth, the analysis focuses on the interplay between the network structure of intermediate input use - the structure of the input matrix, Γ- and the propagation of sectoral shocks, as given by the equilibrium expression (8).

The following Lemma is key for the rest of the analysis in that it yields a simple factorization of the input-use matrix Γ into the product of two square matrices: a binary adjacency matrix $A(G)$, giving the structure of intersectoral linkages in the economy - defining who trades with whom - and a diagonal matrix D_γ setting the scale of input transactions between two sectors by defining the level of the cost shares for the non-zero elements of $\Gamma(G)$.

Lemma 1 *Assume that, for each sector $j = 1, ..., M$, $\gamma_{ij} = \gamma_{kj}$, for all inputs i and k necessary to the production of output in sector j, that is for all pairs $\gamma_{ij}, \gamma_{kj} \neq 0$. Then, the input-use matrix Γ is given by:*

$$\Gamma(G) = A(G)D_\gamma$$

where $A(G)$ is a binary adjacency matrix representation of the intersectoral network, and D_γ is a diagonal matrix with a typical element $D_{kk} = \frac{\gamma_k}{d_k^{in}}$, where $\gamma_k < 1$ and $d_k^{in} > 0$ is the indegree of sector i. Further, for any M and any A, the columns sums of $\Gamma(G)$ are given by $\gamma_j < 1$, for $j = 1, .., M$ and $(I - \Gamma(G))^{-1}$ is well defined.

The proof of the Lemma follows immediately from the assumption that $\gamma_{ij} = \gamma_{kj}$ for all $\gamma_{ij}, \gamma_{kj} \neq 0$. This assumption will be used throughout the paper as it simplifies considerably the description of a sectoral technology by imposing homogeneity along the intensive margin of intersectoral trade - necessary inputs for any given sector have a symmetric role- while allowing for substantial heterogeneity along the extensive margin - sectors can differ in the number of sectors they demand inputs from or supply inputs to.

[14] $(I - \Gamma)^{-1}$ exists if every eigenvalue of Γ is less than one in absolute value. From the Frobenius theory of non-negative matrices, the maximal eigenvalue of Γ is bounded above by the largest column sum of Γ, $\max_k \{\gamma_k\}_{k=1}^M$, which is less than one.

[15] In general, technology shocks also have effects on upstream demand, by changing the demand of inputs necessary to produce output and changing sectoral output level. In the current setting, due to the Cobb-Douglas assumption on preferences and technology, these two effects cancel out exactly; see Shea (2002).

3.2 Representing Sectoral Linkages as Networks.

The upshot of the assumption in the previous Lemma is that I need only to specify two objects to be able to define Γ: a binary matrix announcing who supplies whom and a vector giving the cost share of intermediate inputs for each sector. The individual cost shares are then given immediately by the Lemma. In this subsection I show how to construct a data-generating process for random matrices, $\mathcal{A}$, from which individual members - matrices of intersectoral linkages, A - are drawn. This will serve as a device to generate lists of intermediate inputs necessary for the production of each sector. In particular, I show how to encode the empirical characterization of large-scale input flow data put forth in the previous section. I do this by specifying this data-generating process according to three parameters: a parameter controlling the dimension of the problem - given by the number of sectors M; a demand side parameter $\overline{e}$, controlling the average connectivity in the economy - given by the number of inputs the average sector demands - and a supply side parameter ζ, controlling the heterogeneity across sectors in their role of input-suppliers.

To construct this data-generating process $\mathcal{A}(M, \overline{e}, \zeta)$ I develop a simple digraph extension of Chung and Lu's (2002, 2006) model of undirected random graphs with given expected degree sequences. Thus, I will be considering realizations of input-supply links (edge sets) in the following way: for a given number of sectors M, associate to the collection of all ordered pairs of sectors/vertices $\{v_i, v_j\}, v_i, v_j \in V$, an array of independent, Bernoulli random variables, A_{ij}, taking values 1 or 0 with probability p_{ij} and $1 - p_{ij}$ respectively. Now define a realization of the intersectoral trade network as an edge set E such that $\{v_i, v_j\}$ is an element of the edge set E, if $X_{ij} = 1$. Notice that I can then compute the expected outdegree of any sector as $E(d_i^{out}) = E(\sum_j A_{ij}). = \sum_j p_{ij}$, given independent realizations of each supply-to link. Similarly the expected in-degree of a sector can be computed as $E(d_i^{in}) = E(\sum_i A_{ij}). = \sum_i p_{ij}$. The remainder of this section shows alternative ways to parameterize these sectoral linkage probabilities p_{ij}.

To achieve this, for a given M, associate a weight sequence $e \doteq \{e_1, ..., e_M\}$ to the collection of sectoral labels, such that $e_i \in [0, M]$. Now, for each possible ordered pair of sectors $\{v_i, v_j\} \in V^2$ define the probability of having a directed arc from $v_i \longrightarrow v_j$ as

$$p_{ij} \doteq \frac{e_i}{M}, \forall j \in V \tag{9}$$

This encodes: *i)* a sector with higher weight, e_i, will have a higher probability to supply every sector in the economy and *ii)* for any given j, the probability of sector i being its input supplier depends only on the label of sector i and is thus not responsive to the label of j[16]. These are a strong assumptions in that when describing whether an input trade relationship exists or not, both the identity of the supplying *and* that of demanding sector -label $j-$ should matter. Effectively, this reduces the problem of how to specify input-linkages for every pair of sectors to a simpler problem of distinguishing sectors by how likely they are to be general purpose suppliers (i.e. sectors that have an e_i close to M). However, its tractability yields two immediate results. First, for a given M, the expected out-degree of a sector i, $E(d_i^{out})$, will be given by:

$$E(d_i^{out}) = \sum_j p_{ij} = e_i, \quad i = 1, ..., M \tag{10}$$

Second, for any sector i, its expected indegree, $E(d_i^{in})$, is given by:

$$E(d_i^{in}) = \sum_i p_{ij} = \frac{\sum_i e_i}{M}, \forall i \tag{11}$$

That is, matrices of intersectoral linkages, A, drawn from the sampling scheme above will yield, on average, as much heterogeneity in sectors along their supply dimension as the modeler feeds it through the weights $\{e_i\}_{i=1}^{M}$. Conversely it will generate homogeneity in terms of the number of sectors a randomly chosen sector buys inputs from, i.e. it yields sectors that will be alike in terms of the number of inputs they demand[17].

What is left is to understand is how to specify the weight sequence $\{e_i\}_{i=1}^{M}$. Two deterministic, and rather extreme cases, serve as a useful starting point. Thus, consider first a setting where each sector is an input supplier to every other sector in the economy. This is isomorphic to *complete* network of sectoral linkages, where for any two pair of vertices there is a directed arc from one to the other with probability one. The weight sequence that generates it is simply $e_i = M$ for all $i = 1, .., M$, and, necessarily, $d_i^{out} = d_i^{in} = M$ for all i. Conversely, an economy where, with probability one, there is sole input supplier maps directly into a *star* network, where one vertex acts as a hub with directed arcs from this vertex to all other vertices. An intermediate case is given by a $N-star$ network, where N out of M sectors in the economy act as material input suppliers to every sector and the

[16] Chung and Lu's (2002, 2006) original model for undirected graphs gives $p_{ij} = \frac{e_i e_j}{\sum_{k=1}^{M} e_k}$ so that $p_{ij} = p_{ji}$ for all i, j. When studying inter-sectoral supply links this symmetry is uncalled for: the fact that sector i has a high probability of supplying to j should not imply the converse.

[17] It is easy to adapt the arguments in Chung and Lu (2006, pp. 100-101) to go further and show that actual (sampled) sequences of sectoral outdegrees will concentrate around its expected value and offer bounds that are tight for the larger sectors (in term of outdegrees).

remaining ones are solely devoted to final goods production. In this case, the corresponding weight and outdegree sequences would be $e_i = d_i^{out} = 0$ for $i = 1, .., M - N$ and $e_i = d_i^{out} = 1$ for $i = M - N + 1, ..., M$, while the indegree sequences would be given by $d_i^{in} = N$ for all $i = 1, ..., M$. The following definition summarizes these two cases in terms of their adjacency matrices:

Definition 5 *For a given number M sectors, i) a complete network of sectoral linkages is represented by a $M \times M$ binary matrix $A(G^C)$ where, for each element $A_{ij}(G^C)$, $Pr(A_{ij}^C(G^C) = 1) = 1$ for all $i, j = 1, ...M$ and ii) an N-star network of sectoral linkages is represented by a $M \times M$ binary matrix, $A(G^S)$, where, for each element $A_{ij}(G^S)$, $Pr(A_{ij}(G^S) = 1) = 0$ for all (i, j) pairs of sectors such that $i = 1, ..., M - N$ and $j = 1, ..., M$ and $Pr(A_{ij} = 1) = 1$ for (i, j) pairs of sectors such that $i = M - N + 1, ..., M$ and $j = 1, ..., M$.*

Given a sequence of cost shares of intermediate inputs $\{\gamma_j\}_{i=1}^M$, I can then form the corresponding input-use matrices, $\Gamma(G^C)$ and $\Gamma(G^S)$ by using the Lemma in the previous subsection.

While providing simple benchmarks, the two networks above are too simple to capture the patterns of sectoral linkages described in the previous section. To achieve this, in the remainder of this subsection, I follow Chung, Lu and Vu (2003) and Chung and Lu (2006), and specify weights e_i such that i) the expected outdegree sequence follows an exact power law sequence and ii) all sectors have the same expected indegree:

Definition 6 *Fix a triplet of parameters $(M, \overline{e}, \zeta)$. Let $\mathcal{A}(M, \overline{e}, \zeta)$ denote a data generating process for power law sectoral linkages, whose draws are $M \times M$ binary matrices $A(G^{PL})$, where for each element $A_{ij}(G^{PL})$, the $Pr(A_{ij}(G^{PL}) = 1) = p_{ij}$ is given by [9] and the weight sequence is given by*

$$e_i = ci^{-\frac{1}{\zeta-1}} \quad \text{for } 1 \leq i \leq M \text{ and } \zeta > 2 \tag{12}$$

and

$$c = \frac{\zeta - 2}{\zeta - 1}\overline{e}M^{\frac{1}{\zeta-1}} \tag{13}$$

To see how this parameterization for link probabilities implies a power law sequence for expected out-degrees, notice that I can use expression (12) to solve for i and get:

$$i \propto E(d_i^{out})^{-\zeta+1} \tag{14}$$

Now suppose I rank sectors according to the expected number of sectors they supply inputs to $E(d_i^{out})$. The expression in (12) implies that they will be ranked according to i:

$i = 1$ giving the largest sector, $i = 2$ the second largest and so forth. Notice also that, by definition, there are i sectors that, in expectation, supply to at least the same number of sectors supplied by the $i^{th}-$largest sector. Thus a sector's rank i is proportional to the fraction of sectors larger than i. What expression (14) is stating is that the log of this fraction will scale linearly with the log expected out-degree of sector i, with parameter ζ controlling the scaling behavior. Thus, the expected outdegree sequence is an exact power law sequence[18].

Notice also that the tail parameter ζ only controls the shape of the outdegree distribution - how fat-tailed the distribution will be - but not the average indegree, which is a free parameter, $\overline{e}$. That is, it is possible to show that, under parameterization (12), the average weight, $\frac{\sum_{i=1}^{M} e_i}{M} \simeq \overline{e}$, .by approximating a finite sum with an integral thus:

$$
\begin{aligned}
E(d_i^{in}) &= \frac{\sum_i^M e_i}{M} = \frac{1}{M}\frac{\zeta-2}{\zeta-1}\overline{e}M^{\frac{1}{\zeta-1}}\sum_{i=1}^{M} i^{-\frac{1}{\zeta-1}} \\
&\simeq \frac{1}{M}\frac{\zeta-2}{\zeta-1}\overline{e}M^{\frac{1}{\zeta-1}}\int_1^M i^{-\frac{1}{\zeta-1}}di \\
&= \overline{e} - o(1), \ i = 1, ..., M
\end{aligned}
$$

Figure 5 below plots the $\mathcal{A}(M, \overline{e}, \zeta)$ model-based equivalent of Figures 3 and 4, the indegree density and the outdegree CCDF. It presents the sectoral demand-supply side breakdown for thirty A matrices drawn at random from a family of intersectoral digraphs, $\mathcal{A}(M, \overline{e}, \zeta)$ where I have picked the following parametrization: M is given by a 500 sector economy, where the average number of inputs needed per sector, $\overline{e}$, is set at 15, and the parameter controlling heterogeneity of sectors along the supply side, ζ, is set at 2.1. This parameterization is based on the corresponding objects computed from the B.E.A. detailed input-use matrices in Section 2.

[18] This is a *deterministic* sequence with power-law like (or scaling) behavior in that it gives a finite sequence of real numbers, $E(d_1^{out}) \geq E(d_2^{out}) \geq ... \geq E(d_M^{out})$, such that $i = c\left[E(d_i^{out})\right]^{-\varphi}$ where c is a constant and φ is called the scaling index. See Li et al (2006) for a useful discussion on scaling sequences vs. power law distributions.

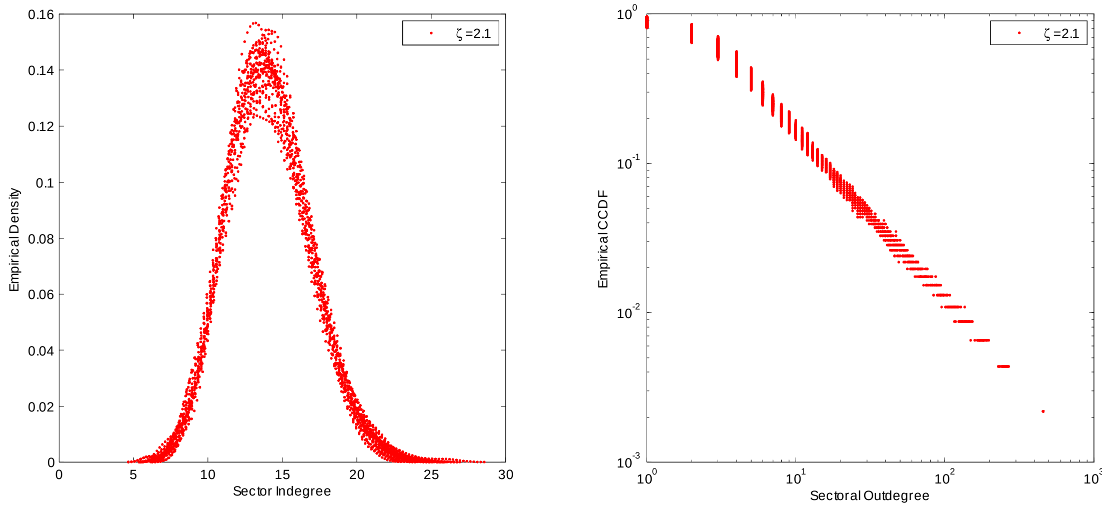

Figure 5: *Empirical indegree density (l.h.s.) and outdegree CCDF (r.h.s.) for 30 intersectoral trade structures drawn at random from* $\mathcal{A}(M,\bar{e},\zeta)$ *for* $M=500$*,* $\bar{e}=15$*,* $\zeta=2.1$.

While individual realizations of A are random objects, thus differing in the exact placement of zeros, the indegree and outdegree sequences implied by each realization of the intersectoral network yield similar patterns. In other words, row and column sums will not differ much across realizations. By design, each realization of $\mathcal{A}(M,\bar{e},\zeta)$, retains the features noted in Section 2: homogeneity along the demand side - sectoral indegrees concentrate along the specified average degree, $\bar{e}$ - and heterogeneity along the supply side, where the number of sectors any given sector supplies can differ by orders of magnitude. Namely, the outdegree sequences implied by realizations of A display fat-tails in the form of a power law- as instructed by Definition 6.

Given a realization of $A(G^{PL})$ and a sequence $\{\gamma_j\}_{i=1}^{M}$ I can again resort to Lemma 1 to form the corresponding input-use matrix $\Gamma(G^{PL})$. However, some care is needed in applying the Lemma, as realizations of A are now random objects. In particular, recall that the Lemma requires that the indegree, d_i^{in}, is strictly positive for all sectors $i=1,...,M$ in order for $(I-\Gamma(G^{PL}))^{-1}$ to be well defined.The following Lemma gives the probability that this is indeed the case under the data generating process $\mathcal{A}(M,\bar{e},\zeta)$.

Lemma 2 *Fix a triplet of parameters* $(M,\bar{e},\zeta)$*. Let* $\{d_1^{in},...,d_M^{in}\}$ *denote the sampled indegree sequence, associated to a realization of* $A(G^{PL})$ *under the data generating process* $A(M,\bar{e},\zeta)$*. Then, with probability* $\left[1-\Pi_{j=1}^{M}\left(1-\frac{e_j}{M}\right)\right]^M$*, all elements of the indegree sequence* $\{d_1^{in},...,d_M^{in}\}$ *are strictly positive.*

Thus, and noticing that given a triplet $(M, \overline{e}, \zeta)$, I can compute the weight sequence $\{e_j\}_{j=1}^{M}$, I can always compute this probability. For example, for the data generating process $\mathcal{A}(500, 15, 2.1)$ considered above, the probability that $(I - \Gamma(G))^{-1}$ exists is 0.996. More generally, for a fixed M, and given the statement in Definition 6, the larger is their expected indegree, the smaller is the probability that I sample sectors demanding no intermediate inputs. Alternatively, fixing M and $\overline{e}$, the smaller is ζ the higher is this probability since the largest elements of the weight sequence will be closer to M (thus rendering the product term closer to zero). With this technical proviso in mind, I now turn to derive analytical expressions for aggregate volatility as a function of the network structure of intersectoral trade under three different cases: complete, star and power law intersectoral networks.

3.3 Volatility Decay in Sectoral Networks

In this section I return to the static multisector setup put forth in Section 3.1 and show how the structure of intersectoral linkages influences the volatility of aggregate output. For analyzing the latter, and keeping in line with the literature (see Horvath, 1998, or Dupor, 1999), I take the variance of average log output

$$\sigma_Y^2 \equiv E\left[\frac{\sum_{i=1}^{M}(y_i - \mu_i)}{M}\right]^2 \tag{15}$$

as the aggregate volatility statistic. Note that $\sum_{i=1}^{M}(y_i - \mu_i)$ is the sum of log sectoral output (demeaned). Dividing this by the number of sectors gives a log-linear approximation to the more obvious aggregate statistic, the log of total output. The difficulty with the latter is that it involves a nonlinear function of the vector of shocks. The average of log sectoral output can therefore be taken as the log-linearization of this function. Using this aggregate statistic will allow me to compare my results directly with those in Horvath (1998) and Dupor (1999)[19].

Using the tools developed in the previous subsection in tandem with Lemma 1, I can now represent the input-use matrix, Γ, as a function of the network of intersectoral linkages, $\Gamma(G)$, and thus make explicit the link between the latter and aggregate volatility,$\sigma_Y^2(\Gamma(G))$. Propo-

[19] In subsequent work within the same multisector setup presented, Acemoglu et al (2010), show that if we assume i) $\beta_j = \beta$ for all j, ii) $\theta_j = 1/M$ for all j and iii) premultiply the household's utility function by an appropriate normalization constant, then aggregate real value added is given by $\frac{\beta}{M}(I-\Gamma)^{-1'}\boldsymbol{\varepsilon}$. Notice that this is proportional to $\frac{\sum_{i=1}^{M}(y_i-\mu_i)}{M} = \frac{1}{M}(I-\Gamma)^{-1'}\boldsymbol{\varepsilon}$. Thus, under these assumptions, the aggregate statistic σ_Y^2 is proportional to the variance of aggregate real value added.

sition 1 below gives an expression for this statistic in complete network settings, $\sigma_Y^2(\Gamma(G^C)$ versus that obtained with $N-$star networks, $\sigma_Y^2(\Gamma(G^S))$.

Proposition 1 *Assume that the share of material inputs, $\gamma_j = \gamma$, and that sectoral volatility $\sigma_j^2 = \sigma^2$ for all sectors $j = 1, ..., M$. Consider the equilibrium of a static multisector economy (8) where the input-use matrix, Γ, is given by $\Gamma(G^C)$ or by $\Gamma(G^S)$. In either case, $(I-\Gamma)^{-1}$ is given by*

$$(I-\Gamma)^{-1} = I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'_M$$

where $\boldsymbol{\phi}$ is an $M \times 1$ vector with typical element $\frac{d_i^{out}}{\sum_{i=1}^M d_i^{out}}$ and $\mathbf{1}_M$ is the unit vector of dimension $M \times 1$. Further, aggregate volatility, σ_Y^2 is given by:

$$\sigma_Y^2(\Gamma(G^C)) = \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M} \tag{16}$$

for any complete network of sectoral linkages, and

$$\sigma_Y^2(\Gamma(G^S)) = \left(\frac{N}{M} + \frac{2\gamma}{1-\gamma}\right)\frac{\sigma^2}{M} + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{N} \tag{17}$$

for any N-star network of sectoral linkages.

Notice that with the additional assumptions imposed in the proposition, sectoral technologies in these economies are symmetrical in all respects except, possibly, that some supply to more sectors than others. This is borne out in the expressions for aggregate volatility: they depend only on the share of material inputs, γ, sectoral volatility,σ^2, and the number of effective input suppliers in each case, M or N. The first two effects are standard. Thus, the higher the share of material inputs in production the more aggregate volatility will be affected by disturbances working through the input-output network[20]. Similarly, greater sectoral volatility translates mechanically into heightened volatility in aggregates.

Of interest to this paper is the dependence of aggregate volatility on the number of sectors. Thus, the expression for complete intersectoral structures of input trade is a particular case of the results in Dupor (1999): aggregate volatility scales with $1/M$. To understand how effective the shock diversification argument is in this case notice the following: holding sectoral productivity variance fixed as I move from a five sector economy to a five hundred sector economy, aggregate volatility will be a hundred times smaller. From this, Dupor

[20]This multiplier effect of $(1/1-\gamma)$ on aggregates is a standard feature of multisector economies; see for example the discussion in Jones (2010a, 2010b)

(1999) concludes that the input-output matrix provides a poor propagation mechanism for independent sectoral shocks.

The result for $N-$star sectoral networks offers a different, if somewhat predictable, view. In an economy where the effective number of input suppliers is small, the law of large numbers will be postponed relative to that of Dupor (1999): aggregate volatility now scales with $1/N$, the slowest decaying term in expression (29) [21]. This is Horvath's (1998) argument: limited sectoral interaction yields greater aggregate volatility from sector specific shocks. The difficulty with this result is that the modeler is now left to specify, for each M, what is the number of input suppliers in an economy, N. If, as Horvath (1998) argues, that N is of order $\sqrt{M}$, this would yield a ten fold decrease in aggregate variability as we move from five to five hundred sectors.

I now show that when we abandon these two extreme cases and instead consider more realistic power law sectoral networks - $\Gamma(G^{PL})$- aggregate volatility decays with $M^{-\upsilon}$, where $\upsilon \in (0,1]$ depending on the specific value of the tail parameter in the power law. Thus I show that the power law specification subsumes the two extreme cases above.

Proposition 2 *Assume that the share of material inputs, $\gamma_j = \gamma$, and that sectoral volatility $\sigma_j^2 = \sigma^2$ for all sectors $j = 1, ..., M$. Consider the equilibrium of a static multisector economy (8) where Γ is given by $\Gamma(G^{PL})$ for any $A(G^{PL})$ sampled from the family of input-use graphs $A(M, \overline{e}, \zeta)$. Then, with probability $\left[1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)\right]^M$, $\left[I - \Gamma(G^{PL})\right]^{-1}$ is well defined and given by:*

$$\left[I - \Gamma(G^{PL})\right]^{-1} = I + \frac{\gamma}{1-\gamma}\widetilde{\phi}1'_M + \Theta$$

where $\widetilde{\boldsymbol{\phi}}$ is an $M \times 1$ vector with typical element $\frac{E(d_i^{out})}{\sum_{i=1}^{M} E(d_i^{out})}$ and Θ is an $M \times M$ random matrix with zero column sums. Further, whenever this is the case, $\sigma_Y^2(\Gamma(G^{PL}))$ bounded below by:

$$(1 - o(1))\left(\tfrac{\gamma}{1-\gamma}\right)^2 \kappa_1(\zeta)\tfrac{\sigma^2}{M} \quad \text{if} \quad \zeta > 3$$

or

$$(1 - o(1))\left(\tfrac{\gamma}{1-\gamma}\right)^2 \kappa_2(\zeta)\left[\tfrac{1}{M}\right]^{\frac{2\zeta-4}{\zeta-1}}\sigma^2 \quad \text{if} \quad \zeta \in (2,3)$$

where $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$, positive constants given a ζ.

To interpret the Proposition consider the following thought experiment. Fix a number of sectors, M, and define a typical production technology by setting the average number of

[21] As it should be, the two expressions in Proposition 1 will be equal for $N = M$. Notice that if N is fixed for any M the law of large numbers breaks down completely.

inputs ($\overline{e}$) a sector needs, in order to produce its output. Now entertain two different values of the tail parameter governing heterogeneity across sectors in their role as input suppliers, ζ_1 and ζ_2 such that $2 < \zeta_1 < 3 < \zeta_2$. What this yields are two economies where sectoral production technologies differ in their degree of diversification. Thus ζ_1 economies will be less diversified in that more mass at the tail implies that a greater number of sectors rely on the same general purpose inputs. Conversely, ζ_2 economies, by having more mass at the center of the distribution of input supply links, will be more diversified: there will be a smaller number of hub-like sectors connecting all sectors in the economy and a greater number of specialized input suppliers, each supplying inputs to a smaller fraction of sectors.

The proposition states that the scaling of the aggregate volatility statistic with M is dependent which on region of the parameter space ζ is set, or alternatively, how diversified is the structure of intersectoral linkages in the economy. Thus, for thin tailed distributions of sectoral outdegrees $\zeta > 3$, aggregate volatility scales with the usual term of order $O(1/M)$. This means that the discussion regarding the decay rate in the special case of complete network structures assumed by Dupor, applies also to the current context. Intuitively, in economies with a large number of sectors that do not differ much in their role as input suppliers, aggregate volatility will be negligible.

However, once we consider the fat-tailed region for $\zeta \in (2, 3)$ the decay behavior is altered: the aggregate volatility statistic now decays with M at a rate that is lowered significantly as we consider input use matrices from more heterogeneous outdegree economies.Namely, Proposition 2 yields an analytical expression where the rate of decay in the volatility of aggregate output depends negatively on the degree of fat-tailness in the distribution of sectoral input-supply links. To see this notice that for $\zeta \in (2, 3)$, the term in the expression decays with $M^{-\upsilon}$ where $\upsilon \equiv \frac{2\zeta-4}{\zeta-1} \in (0, 1)$ Namely, as ζ approaches its lower bound of 2, aggregate volatility, $\sigma^2_Y(\Gamma(G^{PL}))$ will converge to zero arbitrarily slower. Taking, for example, the average value of ζ of 2.1. in Section 2, yields a much slower decay of order $\sqrt[6]{M}$ or $\sigma^2_Y(\overline{\Gamma}) \propto \frac{\sigma^2}{\sqrt[6]{M}}$. To have an idea of the magnitudes involved, this means that as I move from, say, a five sector economy to a five hundred sector economy I expect to find only a *two-fold* decrease in aggregate volatility. Thus, strong heterogeneity across input-supplying sectors opens the possibility of generating non-negligible aggregate fluctuations even in large scale multi-sectoral contexts[22].

In short higher values of ζ yield greater technological diversification: sectoral technologies

[22]Notice also that Horvath (1998) conjecture of a $\sqrt{M}$ decay in aggregate volatility is obtained by fixing ζ at a very particular point: $\zeta = 2.333$.

are relatively more reliant on specialized input-suppliers and less so on common, general purpose, inputs. Therefore, greater diversification in the form of less reliance on common inputs will yield only loosely coupled technologies and, as a result, lower aggregate volatility. Less diversification induces strongly coupled technologies and thus a stronger propagation mechanism[23]. The next section will show that this intuition carries through when we move to dynamic multi-sector settings.

4 Dynamic Multi-Sector Economies

This section recalls a baseline dynamic multi-sectoral model, as introduced in Horvath (1998), Dupor (1999) and Foerster, Sarte and Watson (2008). This is a multi-sector version of a one-sector Brock-Mirman stochastic economy. Following Horvath (1998) and Dupor (1999), I show that, for a particular case where it is possible to solve for the planner's solution analytically, the results derived in the previous section extend to a dynamic setting. I then return to the general setup and present some quantitative explorations.

4.1 General Setup

A representative agent maximizes her expected discounted log utility from infinite vector valued sequences of consumption of M distinct goods and leisure.

$$E_0 \sum_{t=0}^{\infty} \beta^t \left[\sum_{j=1}^{M} \log(C_{jt}) - \psi L_{jt} \right] \tag{18}$$

where β is a time discount parameter in the $(0, 1)$ interval, L_{jt} is labor devoted to the production of the j^{th} good at time t. Expectation is taken at time zero with respect to the infinite sequences of productivity levels in each sector, the only source of uncertainty in the economy.

The production technology for each good $j = 1, ..., M$ combines sector-specific capital, labor and intermediate goods in a Cobb-Douglas fashion:

$$Y_{jt} = Z_{jt} K_{jt}^{\alpha_j} L_{jt}^{\varphi_j} \prod_{i=1}^{M} M_{ijt}^{\gamma_{ij}} \tag{19}$$

[23] See Simon and Ando (1961) for a distant forerunner in analyzing the implications of loose vs. strong coupling across units.

where K_{jt}, and Z_{jt} are, respectively, time t, sector j, value of sector specific capital stock and its (neutral) productivity level. M_{ijt} gives the amount of good i used in sector j in period t. Further, define

$$\gamma_j = \sum_{i=1}^{M} \gamma_{ij}$$

with γ_{ij} denoting the cost-share of input from sector i in the total expenditure on intermediate inputs for sector j (allowed to take the value of zero). Again I can arrange the cost shares in a $M \times M$ input-use matrix, Γ. Constant returns to scale are assumed to hold at the sectoral level such that:

$$\alpha_j + \varphi_j + \sum_{i=1}^{M} \gamma_{ij} = 1, \forall_j \tag{20}$$

It's assumed that sector-specific capital depreciates at rate δ :

$$K_{jt+1} = I_{jt} + (1 - \delta) K_{jt} \tag{21}$$

where I_{jt} is the amount of investment in sector $j's$ capital at time t. Note that due to the sector-specific nature of capital, the sectoral resource constraints are given by:

$$Y_{jt} = C_{jt} + K_{jt+1} - (1 - \delta) K_{jt} + \sum_{i=1}^{M} M_{jit} \tag{22}$$

Finally, I further assume that the log of sector specific productivity follows a random walk

$$\ln(Z_{jt}) = \ln(Z_{jt-1}) + \varepsilon_{jt},\ \varepsilon_{jt} \sim N(0, \sigma^2) \tag{23}$$

where the sectoral innovations are assumed to be i.i.d. both in the cross section and across time.

Definition 7 *The Social Planner's problem is to choose sequences of sector specific capital* $\{K_{jt+1}\}_{j,t}$, *intermediate inputs* $\{M_{ijt}\}_{i,j,t}$ *labor* $\{L_{jt}\}_{j,t}$ *and consumption allocations*$\{C_{jt}\}_{j,t}$ *such that, given a vector of time zero capital stocks* $\{K_{j0}\}_j$ *and a sequence of sectoral productivity levels* $\{Z_{jt}\}_t$ *drawn from (23), the following hold true:*

i)$\{C_{jt}, L_{jt}\}_{j,t}$ *maximizes the representative consumer expected lifetime utility given by (18)*

ii) the sectoral resource constraint (22) is satisfied, sector by sector, for all time periods, where Y_{jt} *is given by (19).*

iii) the labor allocation across sectors is feasible, $\sum_{j=1}^{M} L_{jt} = L$ *for all* t*, where* L *is the time endowment of the household*

As Foerster, Sarte and Watson (2008) show, the characterization of the deterministic steady state of this model is analytically tractable and a log-linearization around that steady state yields:

$$\Delta \mathbf{y}_{t+1} = \Phi \Delta \mathbf{y}_t + \Pi \boldsymbol{\varepsilon}_{t+1} + \Xi \boldsymbol{\varepsilon}_t \tag{24}$$

where $\Delta \mathbf{y}_{t+1}$ is an $M \times 1$ vector of percentage deviations around the sectoral steady state, $\boldsymbol{\varepsilon}_{t+1}$ is a vector of sectoral productivity shocks and Φ, Π and Ξ are $M \times M$ matrices that depend on model parameters only.

4.2 Analytical Solutions in a Special Case

I now take a special case of the setup above, where explicit analytical solutions are available. In particular, I follow Horvath (1998) and Dupor (1999) and assume that there is no labor ($\varphi_j = 0$ for all j) and that sector-specific capital depreciates fully ($\delta = 1$). Under these assumptions, Dupor (1993, fn.3) and Foerster, Sarte and Watson (2008) show that the planner's problem now yields an analytical solution given by the first order autoregression:

$$\Delta \mathbf{y}_{t+1} = (I - \Gamma)^{-1\prime} \alpha_d \Delta \mathbf{y}_t + (I - \Gamma)^{-1\prime} \boldsymbol{\varepsilon}_{t+1} \tag{25}$$

where α_d is a $M \times M$ diagonal matrix with the vector of capital shares α on its diagonal. As in the simple static setup of Section 3, it is the Leontieff inverse $(I - \Gamma)^{-1}$ that mediates the propagation of independent technology shocks at the sectoral level. Now, in order to characterize the second moment properties of this economy, I study the spectral density function for sectoral output growth induced by expression (25) above. This is possible since, under the assumptions made here, the $\{\Delta \mathbf{y}_t\}_t$ sequence given by (25) is stationary and thus admits an infinite moving average representation which, in turn, implies a frequency domain representation. In particular, under the assumptions made above, it is easy to show that the population spectrum for sectoral output growth, $\boldsymbol{\Delta} \mathbf{y}_t$, at frequency ω is given by

$$S_{\boldsymbol{\Delta}\mathbf{y}}(\omega, \Gamma) \doteq \frac{\sigma^2}{(2\pi)} (I - \alpha_d e^{-i\omega} - \Gamma')^{-1} (I - \alpha_d e^{i\omega} - \Gamma)^{-1} \tag{26}$$

Furthermore, given an $M \times 1$ vector $\mathbf{w}$ of aggregation weights, the spectrum for aggregate output growth at frequency ω is given by

$$S(\omega, \Gamma) \doteq \mathbf{w}' \mathbf{S}_{\boldsymbol{\Delta}\mathbf{y}}(\boldsymbol{\omega}, \boldsymbol{\Gamma}) \mathbf{w} \tag{27}$$

The spectral density function is a useful object in that it provides a complete characterization of the autocovariance function for average sectoral output growth. Notice that by setting the elements of $\mathbf{w}$ to be equal and given by $1/M$, $S(\omega, \Gamma)$ is gives the dynamic counterpart to the aggregate statistic (15) of the static model of Section 3

I now turn to characterizing the decay of the univariate spectral density expression (27) with the number of sectors for the case of power law sectoral linkages $\Gamma(G^{PL})$.

Proposition 3 *Assume that the share of material inputs, $\gamma_j = \gamma$ for all sectors $j = 1, ..., M$. Consider the population spectrum for sectoral output growth $S_{\mathbf{\Delta y}}(\omega, \Gamma)$ (26) where Γ is given by $\Gamma(G^{PL})$ for any $A(G^{PL})$ sampled from the family of input-use graphs $A(M, \overline{e}, \zeta)$. Then, with probability $\left[1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)\right]^M$, $\left[I - \Gamma(G^{PL})\right]^{-1}$ is well defined. Whenever this is the case and for aggregation weights $\mathbf{w} = (1/M)\mathbf{1}_M$, the spectral density for aggregate output growth $S(\omega, \Gamma(G^{PL}))$, is bounded below by :*

$$\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{\sigma^2}{M} + (1 - o(1))\gamma^2\kappa_1(\zeta)\frac{\sigma^2}{M}\right] \quad \textit{if } \zeta > 3$$

and

$$\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega) - \gamma^2)\frac{\sigma^2}{M} + (1 - o(1))\gamma^2\kappa_2(\zeta)\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\sigma^2\right] \quad \textit{if } \zeta \in (2, 3)$$

where $a(\omega) = \frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$, $b(\omega) = (1 - \alpha e^{i\omega})(1 - \alpha e^{-i\omega})$, $\kappa_1(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta) = \frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

As in Proposition 2, the expression for the volatility of aggregates differs according to the tail parameter governing heterogeneity across sectors in their role as input suppliers. Thus for $\zeta > 3$, i.e. thin tail distributions, or diversified economies, the expression again recovers the strong diversification of shocks argument given in Dupor. Volatility in aggregate variables decays at rate M as we expand the number of sectors, yielding negligible aggregate volatility for any moderate level of disaggregation. Conversely, for economies where large input-supplying hubs form the basis for input trade flows, this decay rate is slowed down arbitrarily as ζ approaches its lower bound. The more every sectoral technology in an economy relies on the same few key inputs the slower the law of large numbers applies.

However, as the Proposition makes clear, this scaling now extends to the autocovariance function of output growth. In particular, exactly the same decay description applies for all frequencies of the spectral density of aggregate output growth. This means that, for any input-use matrix based on sectoral networks given by $\mathcal{A}(M, \overline{e}, \zeta)$, there is a link between volatility and persistence of aggregates and the network structure of the economy. Again, from a network perspective, common reliance on a few input-supplying hubs will induce greater conductance to shocks in those sectors and this in turn generates less subdued and longer lived responses in aggregates.

4.3 Quantitative Explorations in a Network Laboratory

4.3.1 The Spectral Density and its Decay

The data generating process for input use matrices developed in this paper offers a way to model the extensive margin - who trades with whom - of intersectoral trade. In particular, it enables us to generate artificial multisector economies with strong heterogeneity across sectors in their role as input suppliers, while still imposing homogeneity in the number of inputs required for production in each sector. However, recalling the decomposition in Lemma 1, this was achieved at a cost: the model shuts down heterogeneity along the intensive margin for any given sector, by imposing that all inputs used in that sector have equal cost shares.

In contrast, by looking at the detailed input-use data, we can also observe that the intensive margin does indeed play a role: sectors mix inputs in different proportions, thus relying on some more than others in their production activities. The fact that the model for input-use matrices cannot generate such heterogeneity in cost shares might therefore be biasing the results. Notice that, a priori, it is not clear what the direction of this bias should be. If all sectors rely relatively more on the same input then aggregate volatility might well be higher than the one predicted by the model above. Conversely, if different sectors rely heavily on different inputs, the model above should be overpredicting aggregate volatility. It is therefore important to evaluate whether this strong restriction is biasing the results on the volatility of aggregates.

To this effect, I simulate the model implied spectral density of aggregate growth rates by generating input-use matrices artificially. Thus, I will be drawing sectoral linkages networks from $\mathcal{A}(M, \overline{e}, \zeta)$ and then constructing input-use matrices according to Lemma 1. Specifically, the simulations below trace the spectral density for an economy with 523 sectors ($M = 523$) where the average sector demands inputs from fifteen other sectors ($\overline{e} = 15$). Given the power law characterization in Section 2, I choose a tail parameter of $\zeta = 2.1$ to control heterogeneity across input-supplying sectors. As stated above there are constant returns to scale for all sectors, and I further assume that $\sum_{i=1}^{M} \gamma_{ij} = 0.5$ and $\alpha_j = 0.2$ for all sectors.

I set β, the time discount factor to 0.96, δ, the depreciation rate to 0.06 and $\psi = 1$. The aggregation vector $\mathbf{w}$ is constructed from the model-implied steady state shares of sectoral gross output. I repeat this procedure 100 times and report 90% confidence bands for the simulated spectral density. I then compare with the spectrum obtained by using the 1977 detailed IO data[24], on which the parameters of $\mathcal{A}$ are based. Throughout this subsection I keep the standard deviation of the TFP innovation equal to 1.

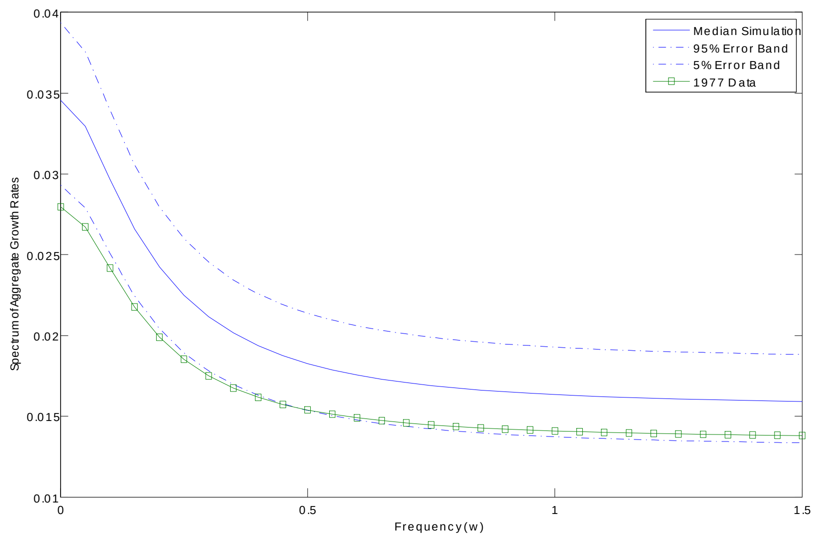

Figure 6: *Data and model-implied spectral densities.*

Figure 6 displays the resulting spectral densities. Notice that the median spectral density of the simulated model economy is larger than that induced by data. This indicates that the intensive margin of trade does give further opportunities for the diversification of sectoral technologies. Thus, the model seems to be somewhat overstating the possibilities for sizeable aggregate fluctuations.

Notice also that, following the same parameterization, I can compute the model-implied spectral density for different levels of aggregation of input-use data. From here, I can then compute what is the rate of decay of the spectral density with the number of sectors and compare it to the predictions on the rate of decay given in Proposition 3. To implement this, I use the input-use data for 1977 available from the BEA at three different levels of aggregation: 523 sectors (mostly 4 digit SIC), 366 sectors (3 digit SIC) and 77 sectors (2 digit SIC). From the latter, I aggregate manually to construct a 36 sector input-use matrix according to the definitions of Jorgenson et al. (2005).

[24] To clarify, I take the original 1977 input use matrix and use the actual shares of intermediate inputs, capital and labor for each of the 523 sectors in the IO data. The aggregation vector is then constructed with the model-implied steady state shares of sectoral gross output under the above parameterization.

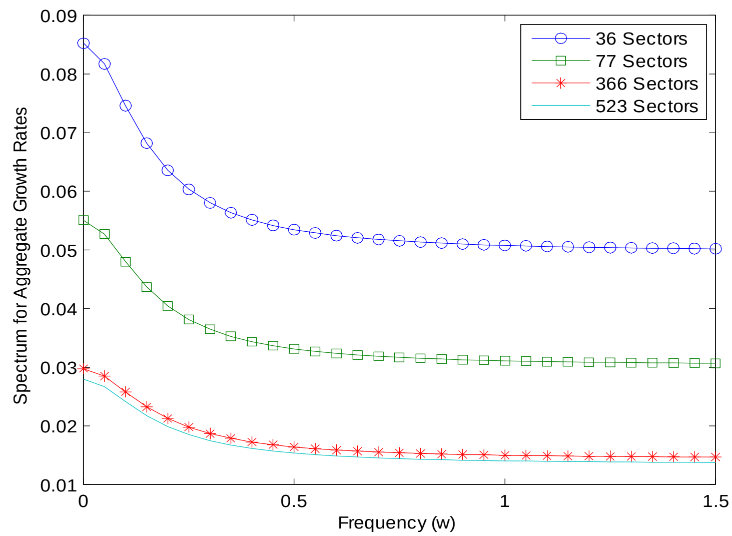

Figure 7: *Spectral Densities for the 1977 input use data across different aggregation levels.*

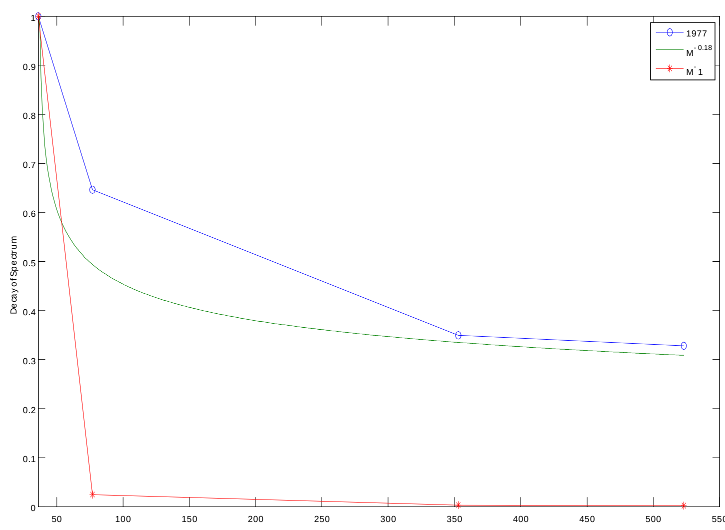

Figure 8: *Spectral density decay for different levels of aggregation of 1977 input-use data at 0.1 frequency.*

Figure 7 plots the spectral density (27) using the 1977 data at various levels of disaggregation. Figure 8 plots the rate of decline in the spectrum at a particular frequency, $\omega = 0.1$. The results are identical for all frequencies. Figure 7 clearly shows that the spectrum of aggregate capital declines with disaggregation. However, Figure 8 reveals that the rate of decrease in the spectrum is slower than that implied by standard law of large numbers ($1/M$ line in the graph). Figure 8 also depicts the rate of decay $M^{-\upsilon}$, predicted by Proposition 3. In particular, I set $\upsilon = 0.18$, obtained by substituting the baseline estimate of $\zeta = 2.1$ in the expression $\upsilon \equiv \frac{2\zeta-4}{\zeta-1}$. Overall, this predicted rate of decay seems to approximate well the actual rate of decay in the 1977 data.

4.3.2 Aggregate Fluctuations and Comovement.

I now assess the performance of aggregates in the multisector i.i.d. setup and compare it to that implied by a panel of detailed sectoral data. In particular, I use the NBER-CES manufacturing industry database to obtain the annual standard deviation of sectoral TFP. Averaging over the 458 4-digit SIC sectors for the sample period 1958-1996 yields an average standard deviation of 7.5. I also back out the implied standard deviation of aggregate gross output growth (by aggregating sectoral growth rates according to their gross output weights) and the average correlation of sectoral output growth with aggregate growth.

I compare these data moments with those implied by the calibrated model above where again I draw input-use matrices from families of matrices $\mathcal{A}(M, \bar{e}, \zeta)$ constructed according to Lemma 1. Specifically, to match the NBER-CES data set, I now draw input-use matrices with 458 hundred sectors ($M = 458$) where the average sector demands inputs from twenty other sectors ($\bar{e} = 15$). Given the results linking different volatility and persistence in aggregates with different levels of heterogeneity in input-supply links - or diversification of sectoral technologies - I consider sampling from two different families: $\zeta = 2.1$ and $\zeta = 3.1$. I maintain the remaining parameters constant across simulations. For each of these values of ζ I draw 100 input-use matrices, simulate the economy for 1000 periods and aggregate according to the model implied steady state shares to obtain 100 series of annual aggregate growth rates. Throughout, the standard deviation of the TFP process (23) hitting each of these sectors is set at the value of 7.5 found in data. Table 3 summarizes the results.

	NBER-CES Data	Model ($\zeta = 2.1$)	Model ($\zeta = 3.1$)
Standard Deviation of Sectoral TFP	7.5	7.5	7.5
Standard Deviation of Aggregate Growth Rate	4.5	2.9	1.2
Average Correlation of Sectoral Output Growth with Aggregate Growth	0.36	0.14	0.06
Number of Common Factors in Sectoral Growth Rates	1	1	0

Table 2. *Selected Moments from Data and Model Calibrations*

The second line of the table above shows that a standard multisector i.i.d. economy with a reasonable variability in sectoral TFP shocks can generate aggregate growth rates that are two thirds as volatile as those seen in data. Recall that a standard LLN reasoning would imply a standard deviation of aggregate growth that is one order of magnitude lower ($7.5/\sqrt{458} = 0.35$), less than one tenth of the observed standard deviation in data. In fact, as we move to a more diversified economy ($\zeta = 3.1$), the implied standard deviation of aggregate growth rates is already 60% lower than that of the reference model and 75% smaller than what we see in data. The $\zeta = 2.1$ model economy is also able to reproduce 40% of the observed average correlation between sectoral output growth and aggregate growth.

Interestingly, while the implied comovement/cross-sectional correlation is smaller than that found in data, it would nevertheless be sufficiently high to induce an outside observer to entertain a 1 shock representation for the panel of sectoral growth rates. The last line in the table reports the outcome of Bai and Ng (2002) common factor tests where I have implemented the PC_{p1} and PC_{p1} estimators both for the NBER panel of sectoral growth rates and for the calibrated multisector economy. Both the data and the $\zeta = 2.1$ model economy yield a 1 factor representation of data while in the counterfactual diversified economy ($\zeta = 3.1$) the Bai and Ng (2002) tests fail to identify any common factor.

Finally, the analysis above suggests that a shock to a large input-supplying sector, i.e. to a general purpose technology, will propagate throughout the economy as a large fraction of other technologies are dependent on it. This means that the structure of intermediate input trade renders the economy *vulnerable* to disturbances in particular sectors. However, one can also proceed to ask a related question: what is the response of the aggregate economy to an average shock? Here I translate an average shock as a shock to an average sector in terms of the number of sectors that it supplies inputs to. Intuition would indicate that the impact of this should be muted by the very fact that the output of an average sector is specialized and demanded only by a limited number of sectors. This in turn generates limited conductance to average shocks. This offers an alternative characterization of the structure of the economy as *robust* to typical shocks.

A simple impulse response analysis illustrates these ideas. Having drawn an input-use matrix, I simulate the growth rate response for each of the four hundred and fifty eight sectors to a one-standard deviation negative shock in the productivity of the largest input-supplying sector (i.e. the sector corresponding to the largest row sum of the sampled A matrix). I then aggregate to see what this implies for the aggregate growth rate response in these economies. I then follow the exact same procedure but instead give a minus one standard deviation pulse to an average sector. That is, for a sampled A matrix, I pick a sector that supplies to fifteen other sectors. If none is found I pick the next largest sector. If more than one is found I shock at random one of the average degree sectors. Figure 9 and 10 below display the outcome of such experiments by displaying the median response over 100 such simulations.

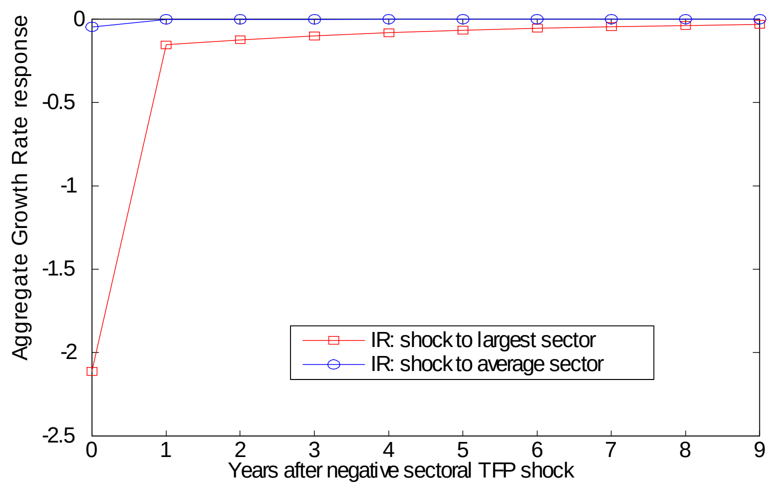

Figure 9: *Impulse response of aggregate growth rate of output to one standard deviation TFP shock to largest and average input suppliers in a* $\zeta = 2.1$ *economy.*

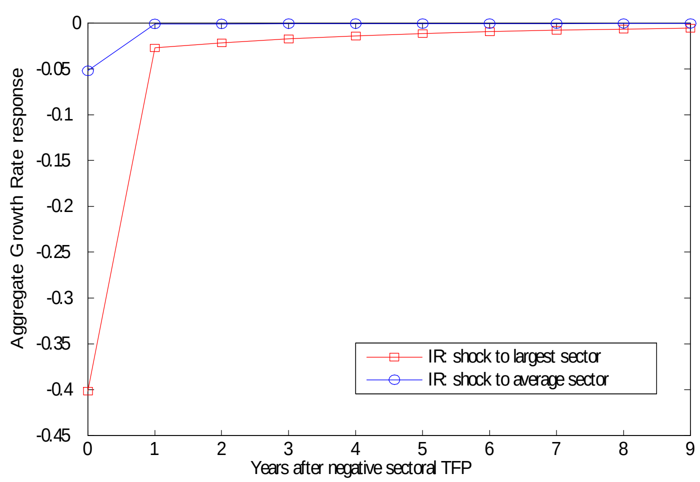

Figure 10: *Impulse response of aggregate growth rate of output to one standard deviation TFP shock to largest and average input suppliers in a* $\zeta = 3.1$ *economy.*

Figure 9 displays precisely the robust-yet-vulnerable nature of economies with limited technological diversification ($\zeta = 2.1$). A shock to the largest sector induces broad comovement in the economy as disturbances in the production technology of a general purpose sector propagate to all sectors in the economy. This yields, on impact, a 2% contraction in aggregate output. In contrast, a shock to an average connectivity sector induces responses in a small number of sectors. Its limited number of connections implies no synchronized movement and as a consequence, propagation to aggregates is weak with aggregate output contracting by 0.05%.

Figure 10 shows what happens when I sample from more diversified economies ($\zeta = 3.1$). The upshot of a thinner tail is that the largest sector sampled from a $\mathcal{A}(458, 15, 3.1)$ family will supply to a relatively smaller number of sectors: as such, propagation is weaker and the mean growth rate response is smaller by one order of magnitude. Interestingly, no such contrast obtains when I consider a shock to an average sector. This suggests that the difference between more and less diversified economies lies in their vulnerability to disturbances in large sectors and not in their robustness to an average shock.

5 Conclusion

Narrowly defined, the starting point of this paper was based on the following insight: setting cost shares to zero for particular intermediate inputs is tantamount to assuming particular network structures for sectoral linkages. From this, I have shown that it is possible to start characterizing sparseness in large-scale input-output data by using a network approach. More importantly, I have built models of intersectoral linkages that retain the first-order connectivity characteristics of data. With this apparatus in hand, the paper employed these tools to solve a controversial question in the business cycle literature: can large-scale multi-sector models with independent productivity shocks generate non-negligible fluctuations in aggregates?

The answer that emerges from this paper is: yes, provided most sectors resort in large measure to the same general purpose inputs. In other words, aggregate fluctuations obtain in economies that are not too diversified in terms of the inputs required by different technologies. Further, input-output data seems to confirm that this is indeed the case, as most sectors rely on key, basic, technologies: oil, electricity, iron and steel, real estate, truck transportation and telecommunications. Sectors are therefore interconnected by their joint reliance on a limited number of general purpose technologies and differ only in the mix of remaining inputs each uses to produce its good.

From a network perspective this means that the linkage structure in the economy is dominated by a few sectoral hubs, supplying inputs to many different sectors. In this case, productivity fluctuations in these hub-like sectors propagate through the economy and affect aggregates, much in the same way as a shutdown at a major airport has a disruptive impact on all scheduled flights throughout a country. In either case, there are no close substitutes and every user is affected by disturbances at the source.

Once one starts to think about the fabric of input-trade in this way, other questions follow suit: can one characterize diversification in networks of sectoral technologies over time or across countries? Take, for example, a problem that has generated recent interest among macroeconomists: the decline in business cycle volatility over the past half century. The conjecture that follows from this paper is that reliance on traditional hubs must have diminished as more specialized substitutes develop. The response of the U.S. economy to past and present oil shocks seems to confirm this view: as alternative energy technologies develop and sectors diversify in their most preferred energy source, the role of oil as a hub to the economy has diminished. As such, oil shocks would likely have a smaller impact on aggregates. Concurrently, the I.T. revolution can be seen as having provided a wealth of alternatives to traditional means of communication and points of sale. The same network perspective can be taken across countries: do less developed economies rely relatively more on a limited number of key technologies? In this sense, can their technologies be characterized as less diversified? If so, the arguments in this paper would predict that less developed economies display more pronounced movements in aggregate output, as indeed seems to be the case in data.

To go beyond these conjectures necessarily implies more careful measurement of the network properties of input-use data and, most likely, more disaggregated data. Indeed, the particular network properties chosen in this paper - tail properties of degree sequences - are both hard to measure and special in that they pertain only to local features of a network. Other measures of connectivity exist and can be of use in characterizing properties of intersectoral trade flows.

At the same time, once one recognizes that network structure is linked to macroeconomic outcomes a more ambitious question emerges: what determines these structures? This requires developing a causal mechanism, i.e. a theory where the network of input-flows is the endogenous outcome of a well-specified economic model. Such theory is surely necessary if one is to think rigorously about the dynamic evolution of these complex objects. This paper falls short of this and makes the easier point that network structure matters. As such, this paper is a starting point for a larger research agenda linking macroeconomic outcomes to the networked structure of modern economies.

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A Sensitivity Analysis for Section 2

In this section I present the results of a sensitivity analysis with regard to the cut-off rule used to define an intersectoral supply link. In particular, I recalculate Table 1 in Section 2, by considering linkages to be defined by transactions that exceed 0.5% or 2% of the total input purchases of that sector (rather than the 1% baseline case). The following tables summarize the results, where I now also include information about the average degree ($\overline{d}$)[25]. Considering first the 2% cutoff rule:

	1972	1977	1982	1987	1992	1997	2002
$\widehat{\zeta}$	2.23	2.19	2.24	2.10	2.20	2.07	2.29
$s.e.(\widehat{\zeta})$	0.19	0.17	0.19	0.13	0.17	0.18	0.19
$\overline{d}$	9	9	9	9	9	9	9
N_tail	71	78	67	93	81	113	78
M	483	523	527	509	478	474	417

Table A1: *MLE estimates $\widehat{\zeta}$, their standard errors ($s.e.(\widehat{\zeta})$),the average degree ($\overline{d}$), the number of observations used to estimate the tail parameter (N_tail) and the total number of sectors (M) for each year from 1972-2002*

Alternatively, the 0.5% cutoff yields:

	1972	1977	1982	1987	1992	1997	2002
$\widehat{\zeta}$	2.10	2.34	1.96	2.45	1.97	2.01	1.91
$s.e.(\widehat{\zeta})$	0.22	0.27	0.23	0.44	0.18	0.17	0.16
$\overline{d}$	24	23	24	24	21	24	24
N_tail	83	67	155	62	138	116	168
M	483	523	527	509	478	474	417

Table A2: *MLE estimates $\widehat{\zeta}$, their standard errors ($s.e.(\widehat{\zeta})$),the average degree ($\overline{d}$), the number of observations used to estimate the tail parameter (N_tail) and the total number of sectors (M) for each year from 1972-2002.*

The average degree changes in a obvious way: the more demanding (lax) is the definition of a sectoral input linkage - i.e. the higher (lower) the cutoff is- the smaller (higher) is the

[25] Recall that the average indegree and the average outdegree have to coincide in a directed graph. I refer to this quantity as average degree.

average connectivity. The average tail parameter estimate, $\widehat{\zeta}$, also does not change much. Recall that for the 1% cut-off rule the average across years was 2.14. Now, for the 2% case, the mean $\widehat{\zeta}$ is 2.19, while for the 0.5% cutoff it is 2.11. I conclude that the characterization put forth in Section 2 is robust to alternative cutoff rules.

B Proof Appendix

The proofs of this paper make repeated use of known results on the inverse of a sum of two matrices. For convenience, I state the relevant ones here:

A.1. Some results on the inverse of a sum of two matrices.

Lemma A.1. *Let A be a nonsingular $M-$dimensional matrix and let U, B and V be $M \times M$ matrices. Then,*

$$(A + UBV)^{-1} = A^{-1} - (I + A^{-1}UBV)^{-1}A^{-1}UBVA^{-1}$$

Proof: See, for example, Henderson and Searle (1981).

A particular case of this is given by the Bartlett inverse:

Lemma A.2. (Bartlett Inverse). *Let A be a square, invertible, $M-$dimensional matrix and u and v be $M-$dimensional vectors. Then:*

$$(A + uv')^{-1} = A^{-1} - \frac{A^{-1}uv'A^{-1}}{1 + v'A^{-1}u}$$

A.2. Proofs

Lemma 2 *Fix a triplet of parameters $(M, \overline{e}, \zeta)$. Let $\{d_1^{in}, ..., d_M^{in}\}$ denote the sampled indegree sequence, associated to a realization of $A(G^{PL})$ under the data generating process $A(M, \overline{e}, \zeta)$. Then, with probability $\left[1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)\right]^M$, all elements of the indegree sequence $\{d_1^{in}, ..., d_M^{in}\}$ are strictly positive.*

Proof: Consider first the probability that a given sector i has an indegree of zero, i.e. that it demands no inputs:

$$\Pr(d_i^{in} = 0) = \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)$$

Thus, the probability that a given sector demands one intermediate input or more is:

$$\Pr(d_i^{in} \geq 1) = 1 - \Pr(d_i^{in} = 0) = 1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)$$

Notice that given that each input linkage is an i.i.d. draw, the indegree sequences are themselves independent draws. Thus, the probability that all sectors demand at least one intermediate input is given by

$$
\begin{aligned}
\Pr(d_1^{in} &\geq 1, d_2^{in} \geq 1, ..., d_M^{in} \geq 1) = \Pi_{i=1}^{M}\left[1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)\right] \\
&= \left[1 - \Pi_{j=1}^{M}\left(1 - \frac{e_j}{M}\right)\right]^M
\end{aligned}
$$

■

Proposition 1 *Assume that the share of material inputs, $\gamma_j = \gamma$, and that sectoral volatility $\sigma_j^2 = \sigma^2$ for all sectors $j = 1, ..., M$. Consider the equilibrium of a static multisector economy (8) where the input-use matrix, Γ, is given by $\Gamma(G^C)$ or by $\Gamma(G^S)$. Then $(I - \Gamma)^{-1}$ is given by*

$$
(I - \Gamma)^{-1} = I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'_M
$$

where $\boldsymbol{\phi}$ is an $M \times 1$ vector with typical element $\frac{d_i^{out}}{\sum_{i=1}^{M} d_i^{out}}$ and $\mathbf{1}_M$ is the unit vector of dimension $M \times 1$. Further, aggregate volatility, σ_Y^2 is given by:

$$
\sigma_Y^2(\Gamma(G^C)) = \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M} \tag{28}
$$

for any complete network of sectoral linkages, and

$$
\sigma_Y^2(\Gamma(G^S)) = \left(\frac{N}{M} + \frac{2\gamma}{1-\gamma}\right)\frac{\sigma^2}{M} + \left(\frac{\gamma}{1-\gamma}\right)^2 \frac{\sigma^2}{N} \tag{29}
$$

for any N-star network of sectoral linkages.

Proof: Define the vectors u and v such that u is an $M \times 1$ vector with each entry u_i restricted $\in [0, M]$ and v an $M \times 1$ vector where each element v_i is given by $\frac{\gamma}{\sum_i u_i}$. Now notice that if I can show that $\Gamma = uv'$ I can then apply Lemma A.1. to get

$$
(I - \Gamma)^{-1} = (I - uv')^{-1} = I + \frac{uv'}{1-\gamma}
$$

where the last equality follows from $v\prime u = \gamma$ (by construction). Notice also that for the static multisector economy deviations of sectoral output from its mean are then given, in vector form, by:

$$
\mathbf{y} - \boldsymbol{\mu} = (I + \frac{uv'}{1-\gamma})'\boldsymbol{\varepsilon}
$$

The rest of the proof shows what the vectors u and v will be for each of the intersectoral networks considered.

Taking the complete case first and noticing that, in this case the indegree of each sector is equal to M, apply Lemma 1 to :

$$\Gamma(G^C) = \left(\frac{\gamma}{M} A(G^C)\right)$$

where $A(G^C)$ is the adjacency matrix of a complete regular digraph of sectoral supply linkages. Notice that, by definition, the average outdegree is $\left(\sum_i d_i^{out}\right)/M = M$. Hence I can rewrite $\Gamma(G^C)$ as

$$\Gamma(G^C) = \gamma M \mathbf{1}_M \mathbf{1}_M^{'} \left(\frac{1}{\sum_i d_i^{out}}\right)$$

Now let u be given by $M\mathbf{1}_M$ (i.e. the outdegree sequence of G_c) and $v' = \mathbf{1}_M^{'} \left(\frac{\gamma}{\sum_i d_i^{out}}\right)$. Clearly, $\Gamma = uv'$ and therefore, using Bartlett inverse result:

$$(I - \Gamma(G^C))^{-1} = I + \frac{\gamma}{1-\gamma} \boldsymbol{\phi} \mathbf{1}_M'$$

where ϕ is an $M \times 1$ vector with typical element $\frac{d_i^{out}}{\sum_{i=1}^M d_i^{out}}$ and 1_M is the unit vector of dimension $M \times 1$. Now,

$$\begin{aligned}
\sum_{i=1}^M y_i - \mu_i &= \sum_{i=1}^M (1 + M \frac{u_i \gamma}{(1-\gamma)\sum_i d_i^{out}}) \varepsilon_i \\
&= \sum_{i=1}^M (1 + M \frac{M\gamma}{(1-\gamma)M^2}) \varepsilon_i \\
&= \sum_{i=1}^M \frac{1}{(1-\gamma)} \varepsilon_i
\end{aligned}$$

Finally given the assumption of i.i.d. sectoral disturbances, $\sigma_Y^2 \equiv E\left[\frac{\sum_{i=1}^M (y_i - \mu_i)}{M}\right]^2$

$$\sigma_Y^2(\Gamma(G^C)) = \left(\frac{1}{1-\gamma}\right)^2 \frac{\sigma^2}{M}$$

as stated in the Proposition.

Now, for the $N - Star$, the indegree of each sector is N, such that by Lemma 1

$$\Gamma(G^S) = \left(\frac{\gamma}{N} A(G^S)\right)$$

where $A(G^S)$ is the binary matrix defined in the text. Notice that, by definition, the average outdegree is now $\left(\sum_i d_i^{out}\right)/M = MN/M = N$.Without loss of generality, order sectors so that the first $M - N$ vertices are not input suppliers and the remaining N sectors supply inputs to every sector in the economy. Then I can write $\Gamma(G^S)$ as

$$\Gamma(G^S) = \gamma M \begin{bmatrix} \mathbf{0}_{M-N} \\ \mathbf{1}_N \end{bmatrix} \mathbf{1}_M^{'} \left(\frac{1}{\sum_i d_i^{out}}\right)$$

Where 0_{M-N} is an $M-N$ dimensional vector of zeros and 1_N is an N dimensional vector of ones. Now, take the Bartlett inverse, where u is given by $M\begin{bmatrix}\mathbf{0}_{M-N}\\ \mathbf{1}_N\end{bmatrix}$, which is simply the out-degree sequence of the $N-star$ network, and $v'=1_M'\left(\frac{\gamma}{\sum_i d_i^{out}}\right)$. Then, again I can write

$$(I-\Gamma(G^S))^{-1}=I+\frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}_M'$$

where ϕ is an $M\times 1$ vector with typical element $\frac{d_i^{out}}{\sum_{i=1}^M d_i^{out}}$ and 1_M is the unit vector of dimension $M\times 1$. Now,

$$\begin{aligned}\sum_{i=1}^M y_i-\mu_i &= \sum_{i=M-N+1}^M (1+M\frac{u_i\gamma}{(1-\gamma)\sum_i d_i^{out}})\varepsilon_i\\ &= \sum_{i=M-N+1}^M (1+\frac{M^2\gamma}{(1-\gamma)\sum_i d_i^{out}})\varepsilon_i\\ &= \sum_{i=M-N+1}^M (1+\frac{\gamma}{1-\gamma}\frac{M}{N})\varepsilon_i\end{aligned}$$

Then, given the assumption of i.i.d. sectoral disturbances

$$\sigma_Y^2(\Gamma_{Star})=\left(\frac{N}{M}+2\frac{\gamma}{1-\gamma}\right)\frac{\sigma^2}{M}+\left(\frac{\gamma}{1-\gamma}\right)^2\frac{\sigma^2}{N}$$

As stated in the Proposition. ■

PROPOSITION 2 *Assume that the share of material inputs, $\gamma_j=\gamma$, and that sectoral volatility $\sigma_j^2=\sigma^2$ for all sectors $j=1,...,M$. Consider the equilibrium of a static multisector economy (8) where Γ is given by $\Gamma(G^{PL})$ for any $A(G^{PL})$ sampled from the family of input-use graphs $A(M,\overline{e},\zeta)$. Then, with probability $\left[1-\Pi_{j=1}^M\left(1-\frac{e_j}{M}\right)\right]^M$, $\left[I-\Gamma(G^{PL})\right]^{-1}$ is well defined and given by:*

$$\left[I-\Gamma(G^{PL})\right]^{-1}=I+\frac{\gamma}{1-\gamma}\widetilde{\phi}1_M'+\Theta$$

where $\widetilde{\boldsymbol{\phi}}$ is an $M\times 1$ vector with typical element $\frac{E(d_i^{out})}{\sum_{i=1}^M E(d_i^{out})}$ and Θ *is an $M\times M$ random matrix with zero column sums. Further, whenever this is the case, $\sigma_Y^2\left[\Gamma(G^{PL})\right]$ bounded below by:*

$$(1-o(1)\left(\tfrac{\gamma}{1-\gamma}\right)^2\kappa_1(\zeta)\tfrac{\sigma^2}{M}\quad\text{if}\quad\zeta>3$$

or

$$(1-o(1)\left(\frac{\gamma}{1-\gamma}\right)^2\kappa_2(\zeta)\left[\frac{1}{M}\right]^{\frac{2\zeta-4}{\zeta-1}}\sigma^2 \quad \text{if} \quad \zeta\in(2,3)$$

where $\kappa_1(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$, *positive constants given a* ζ*.*

Proof: For ease of exposition, I break the proof into three parts. In Part 1, I prove the statement on $\left[I-\Gamma(G^{PL})\right]^{-1}$. In Part 2, I show how to use this statement to express $\sigma_Y^2\left[\Gamma(G^{PL})\right]$, using the matrix-inversion Lemmas given at the beginning of this appendix. Finally, in Part 3, I show that the stated inequality holds true.

Part 1) First notice that under the assumptions stated, Lemma 1 gives, $\Gamma(G^{PL})=\gamma A(G^{PL})D$ where D is a diagonal matrix with a typical element $D_{kk}=\frac{1}{d_k^{in}}$. According to Lemma 2, all the diagonal elements of D are bounded above by 1, with probability $\left[1-\Pi_{j=1}^{M}\left(1-\frac{e_j}{M}\right)\right]^M$. Whenever this is the case, the maximal eigenvalue of $\Gamma(G^{PL})$ is bounded above by γ and therefore $[I-\Gamma(G^{PL})]^{-1}$ is well defined. For any realization of the intersectoral trade digraph $A(G^{PL})$ that satisfies this condition, notice that I can always express $[I-\gamma A(G)D]^{-1}$ as:

$$[I-\gamma A(G)D]^{-1}=\left\{I-\gamma E(A(G))\overline{D}-\gamma[A(G)D-E(A(G))\overline{D}]\right\}^{-1}$$

where E is the expectation operator and $\overline{D}$ is a diagonal matrix with a typical element $\overline{D_{kk}}=\frac{1}{E(d_k^{in})}$. Now, to apply the formula for the inverse of a sum of matrices in Lemma A.1., let

$$\begin{aligned} I-\gamma E(A(G))\overline{D} &\equiv C \\ -\gamma[A(G)D-E(A(G))\overline{D}] &\equiv U \end{aligned}$$

to express the problem as an inverse of a sum of matrices:

$$[I-\gamma A(G)D]^{-1}=[C+I_M U I_M]^{-1}$$

so that the formula for the inverse in Lemma A.1. yields

$$[I-\gamma A(G)D]^{-1}=C^{-1}-C^{-1}U[I+C^{-1}U]^{-1}C^{-1}$$

Now notice that $E[A(G)]$, is an $M\times M$ matrix whose ij entry is given by $E[A_{ij}(G)]=p_{ij}=\frac{e_i}{M}$ and that $\overline{D_{kk}}=\frac{1}{E(d_k^{in})}=\frac{M}{\sum_{i=1}^{M}e_i}$ for all k. Hence $E(A(G))\overline{D}$ can be expressed as a rank one matrix:

$$C=E(A(G))\overline{D}=\boldsymbol{\phi}\mathbf{1}_M'$$

where $\boldsymbol{\phi} = \left[\frac{e_1}{\sum_{i=1}^{M} e_i}, ..., \frac{e_M}{\sum_{i=1}^{M} e_i}\right]$ and $\mathbf{1}' \equiv [1, ..., 1]$. Thus applying Bartlett formula for C^{-1} yields

$$[I - \gamma\boldsymbol{\phi}\mathbf{1}']^{-1} = I + \frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{1 - \gamma\mathbf{1}'I\boldsymbol{\phi}}$$

Since $\mathbf{1}'\phi$= 1 we get for C^{-1}

$$[I - \gamma\boldsymbol{\phi}\mathbf{1}']^{-1} = I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'$$

Now to solve for $[I + C^{-1}U]^{-1}$ substitute in C^{-1} to get

$$\begin{aligned}[I + C^{-1}U]^{-1} &= \left[I + \left(I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right)U\right]^{-1} \\ &= \left[I + U + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'U\right]^{-1}\end{aligned}$$

Notice that for any realization of $A(G)D$,the matrix $U = -\gamma[A(G)D - E(A(G))\overline{D}]$, will have zero column sums. This is so since, by construction, $A(G)D$ and $E(A(G))\overline{D}$ have the same column sums (and equal to 1 for every column). Hence the difference will yield zero column sums. Thus $\mathbf{1}'U = \mathbf{0}'$ where $\mathbf{0}$ is an $M \times 1$ vector of zeros and $\phi\mathbf{1}'U$ is a $M \times M$ matrix of zeros. This implies that:

$$[I + C^{-1}U]^{-1} = [I + U]^{-1}$$

Collecting results

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - \left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]U[I+U]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$$

This expression can be further simplified by again using the fact that $\phi\mathbf{1}'U$ is a matrix of zeros. Thus:

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - U[I+U]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$$

or

$$[I - \gamma A(G)D]^{-1} = \left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \Theta$$

where $\boldsymbol{\phi} = \left[\frac{e_1}{\sum_{i=1}^{M} e_i}, ..., \frac{e_M}{\sum_{i=1}^{M} e_i}\right] = \left[\frac{E(d_1^{out})}{\sum_{i=1}^{M} E(d_i^{out})}, ..., \frac{E(d_M^{out})}{\sum_{i=1}^{M} E(d_i^{out})}\right]$ and Θ is defined as

$$\Theta \equiv \gamma[A(G)D - E(A(G))\overline{D}][I - \gamma[A(G)D - E(A(G))\overline{D}]]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$$

Now, for the zero column sum claim on Θ recall again that, by construction, $A(G)D$ and $E(A(G))\overline{D}$ have the same column sums. Hence their difference will yield a zero column sum matrix. Finally, notice that premultiplication of a matrix by a zero column sum matrix gives again a zero column sum matrix. Hence Θ will have column sums equal to zero.

Part 2) Recalling the definition of U, I can write $\Theta = -U[I+U]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$. Then the static multisector model gives:

$$\mathbf{y} - \boldsymbol{\mu} = \left[\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \Theta\right]' \boldsymbol{\varepsilon}$$

Thus with $\mathbf{1}' \equiv [1, ..., 1]$

$$\begin{aligned}
\sum_{i=1}^{M} y_i - \mu_i &= \mathbf{1}'\left[\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \Theta\right]' \boldsymbol{\varepsilon} \\
&= \mathbf{1}'I\boldsymbol{\varepsilon} + \frac{\gamma}{1-\gamma}\mathbf{1}'(\mathbf{1}\boldsymbol{\phi})\boldsymbol{\varepsilon} + \mathbf{1}'\Theta'\boldsymbol{\varepsilon} \\
&= \sum_{i=1}^{M}\varepsilon_i + \frac{\gamma}{1-\gamma}M\sum_{i=1}^{M}\phi_i\varepsilon_i + \sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij}\varepsilon_i
\end{aligned}$$

Thus the aggregate statistic $\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i$

$$\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i = \frac{1}{M}\sum_{i=1}^{M}\varepsilon_i + \frac{\gamma}{1-\gamma}\sum_{i=1}^{M}\boldsymbol{\phi}_i\varepsilon_i + \frac{1}{M}\sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij}\varepsilon_i$$

which has expectation zero given independent technology shocks. Now we're interested in $E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i\right)^2\right]$

$$\begin{aligned}
E\left[\left(\frac{1}{M}\sum_{i=1}^{M} y_i - \mu_i\right)^2\right] &= \frac{1}{M^2}\sum_{i=1}^{M}E((\varepsilon_i)^2) + \left(\frac{\gamma}{1-\gamma}\right)^2\sum_{i=1}^{M}\phi_i^2E((\varepsilon_i)^2) + \frac{1}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2 E((\varepsilon_i)^2) \\
&+ \frac{2}{M}\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_iE((\varepsilon_i)^2) + \frac{2}{M}\sum_{i=1}^{M}\sum_{j=1}^{M}\theta_{ij}E((\varepsilon_i)^2) \\
&+ \frac{2}{M}\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)E((\varepsilon_i)^2)
\end{aligned}$$

Notice that $\sum_{i=1}^{M}\sum_{j=1}^{M}\theta_{ij} = \sum_{j=1}^{M}\sum_{i=1}^{M}\theta_{ij} = 0$ (since Θ is a zero column-sum matrix), that

$\sum_{i=1}^{M}\phi_i = 1$ and that $E((\varepsilon_i)^2) = \sigma^2$. Using these facts, I can simplify the expression to:

$$\begin{aligned}\sigma_Y^2\left[\Gamma(G^{PL})\right] &= \left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{1}{M}\sigma^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\\ &\quad+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\end{aligned}$$

Part 3). First, notice that by definition of the vector $\boldsymbol{\phi}$

$$\begin{aligned}\sum_{i=1}^{M}\phi_i^2 &\equiv \left[\sum_{i=1}^{M}e_i^2/\left(\sum_{i=1}^{M}e_i\right)^2\right]\\ &= \frac{1}{M}\frac{\sum_{i=1}^{M}e_i^2/\sum_{i=1}^{M}e_i}{\sum_{i=1}^{M}e_i/M}\\ &= \frac{1}{M}\frac{\widetilde{e}}{\overline{e}}\end{aligned}$$

Where $\widetilde{e}\equiv\sum_{i=1}^{M}e_i^2/\sum_{i=1}^{M}e_i$. Chung and Lu (2006, p.109) show that under a power law weight parameterization, $\widetilde{e}$ is given by:

$$\widetilde{e}=\begin{cases}\overline{e}\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)} & \text{if } \zeta>3\\ \overline{e}^{\zeta-2}\frac{(\zeta-2)^{\zeta-1}m^{3-\zeta}}{(\zeta-1)^{\zeta-2}(3-\zeta)} & \text{if } \zeta\in(2,3)\end{cases}$$

where m is the maximum expected outdegree, e_1. Thus, according to Definition 6 $m = \frac{\zeta-2}{\zeta-1}\overline{e}M^{\frac{1}{\zeta-1}}$. Therefore, the first term in the expression above is given by:

$$\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2=\begin{cases}\left(\frac{\gamma}{1-\gamma}\right)^2\kappa_1(\zeta)\frac{\sigma^2}{M} & \text{if } \zeta>3\\ \left(\frac{\gamma}{1-\gamma}\right)^2\kappa_2(\zeta)\left[\frac{1}{M}\right]^{\frac{2\zeta-4}{\zeta-1}}\sigma^2 & \text{if } \zeta\in(2,3)\end{cases}$$

where $\kappa_1(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

Now, grouping the first and the last two terms in the above expression (given at the end of Part 2 of the Proof) for $\sigma_Y^2\left[\Gamma(G^{PL})\right]$:

$$\begin{aligned}&\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\\ &= \sigma^2\sum_{i=1}^{M}\left[\frac{\gamma}{1-\gamma}\phi_i+\frac{1}{M}\sum_{j=1}^{M}\theta_{ij}\right]^2\end{aligned}$$

Each element of this M sum has to be non-negative and at least for some i, strictly positive since not all $\sum_{j=1}^{M}\theta_{ij}$ can be negative (recall that $\sum_{i=1}^{M}\sum_{j=1}^{M}\theta_{ij} = 0$ since Θ is a zero column sum matrix). Thus, summing over i implies that for any M it has to be the case that:

$$\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2 > -\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)$$

I have shown that the first term on the LHS of this expression always goes to zero. Assume the second term also does so, i.e. $\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2$ is $o(1)$ (otherwise, since this is a strictly positive term, the Law of Large Numbers will break down completely in which case the statement of the proposition is trivially true). Now, there are three possible cases: either $\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2$ is $o\left(\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2\right)$ or $\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2$ is $o\left(\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\right)$ or they both approach zero at the same rate. Whatever is the case, for the inequality to hold for any M, the RHS will have to converge to zero at a faster rate than the slowest moving term in the LHS. Thus, $-\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)$ is either $o\left(\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2\right)$ or $o\left(\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\right)$ respectively.

First, take the case where the RHS of the inequality is $o\left(\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2\right)$. Then,

$$\begin{aligned}
\sigma_Y^2\left[\Gamma(G^{PL})\right] &= \left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{1}{M}\sigma^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)+\\
&\quad +\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\\
&> \left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\\
&> (1+o(1))\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2-o\left(\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2\right)\\
&> (1-o(1))\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2
\end{aligned}$$

Alternatively, take the case where the RHS of the inequality is $o\left(\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\right)$. Then,

$$\begin{aligned}
\sigma_Y^2\left[\Gamma(G^{PL})\right] &= \left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{1}{M}\sigma^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)+ \\
&+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2 \\
&> \left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2+\frac{2}{M}\sigma^2\left(\frac{\gamma}{1-\gamma}\right)\sum_{i=1}^{M}\phi_i\left(\sum_{j=1}^{M}\theta_{ij}\right)+\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2 \\
&> (1+o(1))\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2-o\left(\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2\right) \\
&> (1-o(1))\frac{\sigma^2}{M^2}\sum_{i=1}^{M}\left(\sum_{j=1}^{M}\theta_{ij}\right)^2 \\
&> (1-o(1))\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2
\end{aligned}$$

Recalling the solution to $\left(\frac{\gamma}{1-\gamma}\right)^2\sigma^2\sum_{i=1}^{M}\phi_i^2$ show above, I have thus shown that:

$$\sigma_Y^2\left[\Gamma(G^{PL})\right] > \begin{cases} (1-o(1)\left(\frac{\gamma}{1-\gamma}\right)^2\kappa_1(\zeta)\frac{\sigma^2}{M} & \text{if } \ \zeta>3 \\ (1-o(1)\left(\frac{\gamma}{1-\gamma}\right)^2\kappa_2(\zeta)\left[\frac{1}{M}\right]^{\frac{2\zeta-4}{\zeta-1}}\sigma^2 & \text{if } \ \zeta\in(2,3) \end{cases}$$

as claimed in the Proposition. ■

Proposition 3 *Assume that the share of material inputs, $\gamma_j=\gamma$ for all sectors $j=1,...,M$. Consider the population spectrum for sectoral output growth $S_{\mathbf{\Delta y}}(\omega,\Gamma)$ (26) where Γ is given by $\Gamma(G^{PL})$ for any $A(G^{PL})$ sampled from the family of input-use graphs $A(M,\overline{e},\zeta)$. Then, with probability $\left[1-\Pi_{j=1}^{M}\left(1-\frac{e_j}{M}\right)\right]^M$, $\left[I-\Gamma(G^{PL})\right]^{-1}$ is well defined. Whenever this is the case and for aggregation weights $w=(1/M)1_M$, the spectral density for aggregate output growth $S(\omega,\Gamma(G^{PL}))$, is bounded below by :*

$$\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{\sigma^2}{M}+\gamma^2\kappa_1(\zeta)(1-o(1))\frac{\sigma^2}{M}\right] \ \ \textit{if } \zeta>3$$

and

$$\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{\sigma^2}{M}+\gamma^2\kappa_2(\zeta)(1-o(1))\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\sigma^2\right] \textit{ if } \zeta\in(2,3)$$

where $a(\omega)=\frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$,$b(\omega)=(1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$,.$\kappa_1(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ *and* $\kappa_2(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$.

Proof: The proof of the probability statement on $\left[I-\Gamma(G^{PL})\right]^{-1}$ is exactly the same as in the proof of Proposition 2. To show the statement on $S(\omega,\Gamma(G^{PL}))$, start by letting $\lambda=1-\alpha e^{-i\omega}$. By same argument as in Proposition 2. one can always decompose

$$[(1-\alpha e^{i\omega})I-\Gamma]^{-1}=[\lambda I-\gamma A(G)D]^{-1}=\left\{\lambda I-\gamma E(A(G))\overline{D}-\gamma[A(G)D-E(A(G))\overline{D}]\right\}^{-1}$$

Where, again, D is a diagonal matrix with a typical element $D_{kk}=\frac{1}{d_k^{in}}$ and $\overline{D}$ is a diagonal matrix with a typical element $\overline{D_{kk}}=\frac{1}{E(d_k^{in})}$. Now let

$$\begin{aligned}\lambda I-\gamma E(A(G))\overline{D} &\equiv C_\lambda\\ -\gamma[A(G)D-E(A(G))\overline{D}] &\equiv U\end{aligned}$$

and apply Lemma A1 to get:

$$[\lambda I-\gamma A(G)D]^{-1}=C_\lambda^{-1}-C_\lambda^{-1}U[I+C_\lambda^{-1}U]^{-1}C_\lambda^{-1}$$

Solve C_λ^{-1} through the Barttlet inverse formula

$$\begin{aligned}C_\lambda^{-1} &= \lambda^{-1}I+\frac{\lambda^{-1}\gamma\boldsymbol{\phi}\mathbf{1}'\lambda^{-1}}{1-\gamma\lambda^{-1}\mathbf{1}'\boldsymbol{\phi}}\\ &= \lambda^{-1}\left[I+\frac{\lambda^{-1}\gamma\boldsymbol{\phi}\mathbf{1}'}{1-\gamma\lambda^{-1}\mathbf{1}'\boldsymbol{\phi}}\right]\\ &= \lambda^{-1}\left[I+\frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{\lambda-\gamma\mathbf{1}'\boldsymbol{\phi}}\right]\\ &= \lambda^{-1}\left[I+\frac{\gamma\boldsymbol{\phi}\mathbf{1}'}{\lambda-\gamma}\right]\end{aligned}$$

For $[I+C_\lambda^{-1}U]^{-1}$ substitute in C_λ^{-1} to get

$$\begin{aligned}[I+C_\lambda^{-1}U]^{-1} &= \left[I+\lambda^{-1}\left(I+\frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right)U\right]^{-1}\\ &= \left[I+\lambda^{-1}U+\frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'U\right]^{-1}\end{aligned}$$

where again $\phi\mathbf{1}'U$ is a $M \times M$ matrix of zeros. Thus:

$$[I + C_\lambda^{-1}U]^{-1} = I + \lambda^{-1}U$$

Collecting results

$$\begin{aligned}
[\lambda I - \gamma A(G)D]^{-1} &= C_\lambda^{-1} - C_\lambda^{-1}U[I + C_\lambda^{-1}U]^{-1}C_\lambda^{-1} \\
&= \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - \lambda^{-2}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]U[I + \lambda^{-1}U]^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] \\
&= \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] - \lambda^{-2}U[I + \lambda^{-1}U]^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]
\end{aligned}$$

where the last line uses the fact that $\phi\mathbf{1}'U$ is a $M \times M$ matrix of zeros.

Again defining Θ defining as

$$\Theta \equiv \gamma[A(G)D - E(A(G)D)][I - \gamma[A(G)D - E(A(G)D)]]^{-1}\left[I + \frac{\gamma}{1-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]$$

I can rewrite the expression above as

$$[\lambda I - \gamma A(G)D]^{-1} = \lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta$$

Also, by the exact same argument as in the proof of Proposition 2, the column sums of Θ have to be zero for any realization of $A(G)$.

Now using the formula for the aggregate spectrum I get:

$$\begin{aligned}
S(\omega, \Gamma(G^{PL})) &= \frac{\sigma^2}{2\pi}\frac{1}{M^2}\mathbf{1}'[(1-\alpha e^{i\omega})I - \Gamma]^{-1\prime}[(1-\alpha e^{i\omega})I - \Gamma]^{-1}\mathbf{1} \\
&= \frac{\sigma^2}{2\pi}\frac{1}{M^2}\mathbf{1}'\left[\lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta\right]'\left[\lambda^{-1}\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right] + \lambda^{-2}\Theta\right]\mathbf{1} \\
&= \frac{\sigma^2}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{M^2}\left[\begin{array}{c}
\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
+\mathbf{1}'\lambda^{-1\prime}\Theta'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
+\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\lambda^{-1}\Theta\mathbf{1} \\
+\mathbf{1}'\lambda'^{-1}\lambda^{-1}\Theta'\Theta\mathbf{1}
\end{array}\right]
\end{aligned}$$

Now, term by term, in the expression in square brackets. First,

$$\begin{aligned}
\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} &= \mathbf{1}'\left[I + \frac{\gamma}{\lambda'-\gamma}\mathbf{1}\boldsymbol{\phi}' + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}' + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\mathbf{1}\boldsymbol{\phi}'\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} \\
&= M + M\frac{\gamma}{\lambda'-\gamma}\sum_{i=1}^{M}\phi_i + M\frac{\gamma}{\lambda-\gamma}\sum_{i=1}^{M}\phi_i + \\
&\quad + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}M^2\sum_{i=1}^{M}\phi_i^2
\end{aligned}$$

where λ' is the complex conjugate of λ and is given by $1 - \alpha e^{-i\omega}$.

Second,

$$\begin{aligned}
\mathbf{1}'\lambda^{-1\prime}\Theta'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]\mathbf{1} &= \lambda^{-1\prime}\mathbf{1}'\Theta'\mathbf{1} + \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}\mathbf{1}'\Theta'\boldsymbol{\phi}\mathbf{1}'\mathbf{1} \\
&= \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}M\mathbf{1}'\Theta'\boldsymbol{\phi} \\
&= \lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma}M\sum_j\sum_i\theta_{ij}\phi_i
\end{aligned}$$

Where $\Theta'\mathbf{1}$ is a $M \times 1$ vector of zeros (since the columns of Θ sum to zero), so the first term disappears.

Third, and applying same reasoning,

$$\begin{aligned}
\mathbf{1}'\left[I + \frac{\gamma}{\lambda-\gamma}\boldsymbol{\phi}\mathbf{1}'\right]'\lambda^{-1}\Theta\mathbf{1} = \mathbf{1} &\quad {}'\lambda^{-1}\Theta\mathbf{1} + \lambda^{-1}\mathbf{1}'\frac{\gamma}{\lambda'-\gamma}\mathbf{1}\boldsymbol{\phi}'\Theta\mathbf{1} \\
&= \lambda^{-1}\frac{\gamma}{\lambda'-\gamma}M\boldsymbol{\phi}'\Theta\mathbf{1} \\
&= \lambda^{-1}\frac{\gamma}{\lambda'-\gamma}M\sum_j\sum_i\theta_{ij}\phi_i
\end{aligned}$$

Finally,

$$\mathbf{1}'\lambda'^{-1}\lambda^{-1}\Theta'\Theta\mathbf{1} = \lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2$$

Collecting results:

$$\begin{aligned}
S(\omega,\Gamma(G^{PL})) &= \frac{\sigma^2}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{M^2}\left[\begin{array}{c} M + M\frac{\gamma}{\lambda'-\gamma} + M\frac{\gamma}{\lambda-\gamma} + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}M^2\sum_{i=1}^{M}\phi_i^2 \\ +\left(\lambda^{-1\prime}\frac{\gamma}{\lambda-\gamma} + \lambda^{-1}\frac{\gamma}{\lambda'-\gamma}\right)M\sum_j\sum_i\theta_{ij}\phi_i + \lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2 \end{array}\right] \\
&= \frac{\sigma^2}{2\pi}\lambda'^{-1}\lambda^{-1}\left[\begin{array}{c} \frac{\lambda'\lambda-\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\frac{1}{M} + \frac{\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\sum_{i=1}^{M}\phi_i^2 + \\ \left(\frac{2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2}{(\lambda-\gamma)(\lambda'-\gamma)}\right)\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i + \frac{1}{M^2}\lambda'^{-1}\lambda^{-1}\sum_i\left(\sum_j\theta_{ij}\right)^2 \end{array}\right] \\
&= \frac{\sigma^2}{2\pi}\frac{\lambda'^{-1}\lambda^{-1}}{(\lambda-\gamma)(\lambda'-\gamma)}\left[\begin{array}{c} [\lambda'\lambda-\gamma^2]\frac{1}{M} + \gamma^2\sum_{i=1}^{M}\phi_i^2 + \left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i \\ +\frac{1}{M^2}\lambda'^{-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\sum_i\left(\sum_j\theta_{ij}\right)^2 \end{array}\right]
\end{aligned}$$

To establish the lower bound I follow the same strategy as in the proof of Proposition 2. Again the situation I am interested in ruling out is one where $\frac{1}{M}\sum_j\sum_i\theta_{ij}\phi_i$ is negative and decays at a slower rate than $\sum_{i=1}^{M}\phi_i^2$. To see that this is not the case, start by grouping the last three terms in the expression above thus:

$$\sum_{i=1}^{M}\left[\gamma^2\phi_i^2+\left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\theta_{ij}\phi_i+\frac{1}{M^2}\lambda^{\prime-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j\theta_{ij}\right)^2\right]$$

and notice this is simply the i sum of the product of conjugate pairs indexed by i:

$$\sum_{i=1}^{M}\left[\gamma\phi_i+\frac{1}{M}\lambda^{-1}(\lambda-\gamma)\sum_j\theta_{ij}\right]\left[\gamma\phi_i+\frac{1}{M}\lambda^{-1\prime}(\lambda'-\gamma)\sum_j\theta_{ij}\right]$$

Now we know that the product of conjugate pairs is always real and nonnegative. Hence it must be the case for any M, and each i

$$\gamma^2\phi_i^2+\frac{1}{M^2}\lambda^{\prime-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j\theta_{ij}\right)^2\geq-\left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\theta_{ij}\phi_i$$

Again, notice that for some i, $\sum_j\theta_{ij}>0$ -by virtue of the zero column statement on Θ- in which case the inequality will be strict. Thus summing over i:

$$\sum_{i=1}^{M}\gamma^2\phi_i^2+\frac{1}{M^2}\lambda^{\prime-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\sum_{i=1}^{M}\left(\sum_j\theta_{ij}\right)^2>-\left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_{i=1}^{M}\sum_j\theta_{ij}\phi_i$$

which again implies that the term on the RHS of the inequality cannot decay at a slower rate than LHS. By the same arguments as in the proof of Proposition 2, the LHS is either $(1+o(1))\sum_{i=1}^{M}\gamma^2\phi_i^2$ or, a slower decaying, $(1+o(1))\frac{1}{M^2}\lambda^{\prime-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\sum_{i=1}^{M}\left(\sum_j\theta_{ij}\right)^2$ and thus necessarily

$$\begin{aligned}&\sum_{i=1}^{M}\left[\gamma^2\phi_i^2+\left[2\gamma-(\lambda^{-1\prime}+\lambda^{-1})\gamma^2\right]\frac{1}{M}\sum_j\theta_{ij}\phi_i+\frac{1}{M^2}\lambda^{\prime-1}\lambda^{-1}(\lambda-\gamma)(\lambda'-\gamma)\left(\sum_j\theta_{ij}\right)^2\right]\\ >\ &(1-o(1))\sum_{i=1}^{M}\gamma^2\phi_i^2\end{aligned}$$

Using this in the expression for $S(\omega,\Gamma(G^{PL}))$ it has to be the case that for any M and any frequency ω

$$S(\omega,\Gamma(G^{PL}))>\frac{1}{2\pi}\frac{\lambda^{\prime-1}\lambda^{-1}}{(\lambda-\gamma)(\lambda'-\gamma)}\left[(\lambda'\lambda-\gamma^2)\frac{1}{M}+(1-o(1))\sum_{i=1}^{M}\gamma^2\phi_i^2\right]$$

Substituting in for $\lambda=1-\alpha e^{-i\omega}$ and recalling the expression for $\sum_{i=1}^{M}\phi_i^2$ in Proposition 2, yields:

$$S(\omega,\Gamma(G^{PL}))>\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{\sigma^2}{M}+\gamma^2\kappa_1(\zeta)(1-o(1))\frac{\sigma^2}{M}\right]\ if\ \zeta>3$$

and

$$S(\omega,\Gamma(G^{PL}))>\frac{1}{2\pi}\frac{a(\omega)}{b(\omega)}\left[(b(\omega)-\gamma^2)\frac{\sigma^2}{M}+\gamma^2\kappa_2(\zeta)(1-o(1))\left(\frac{1}{M}\right)^{\frac{2\zeta-4}{\zeta-1}}\sigma^2\right]\ if\ \zeta\in(2,3)$$

where $a(\omega)=\frac{1}{(1-\alpha e^{i\omega}-\gamma)(1-\alpha e^{-i\omega}-\gamma)}$,$b(\omega)=(1-\alpha e^{i\omega})(1-\alpha e^{-i\omega})$,.$\kappa_1(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(\zeta-3)}$ and $\kappa_2(\zeta)=\frac{(\zeta-2)^2}{(\zeta-1)(3-\zeta)}$ as claimed ■